(IN)EQUIVALENCE OF COLOR CODE AND TORIC CODE

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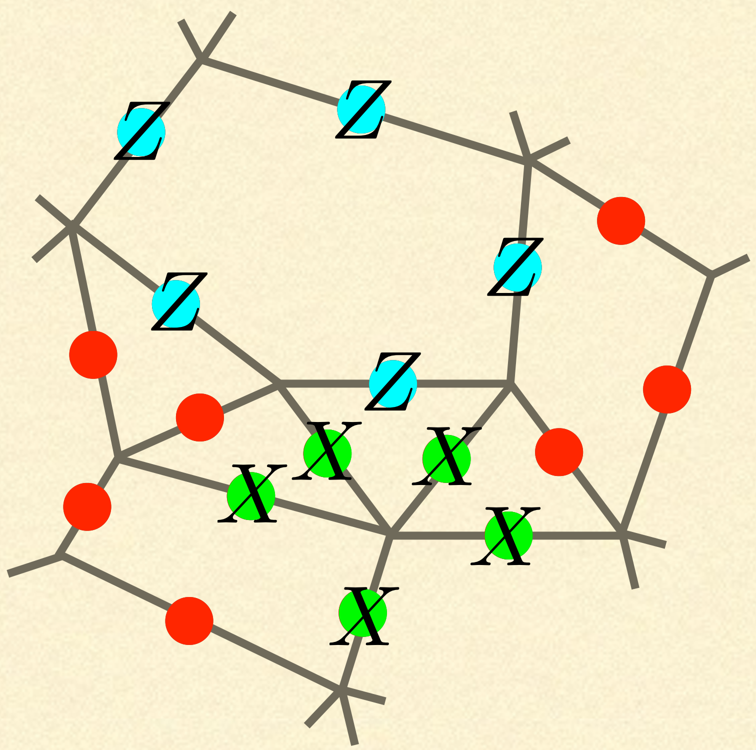
MOTIVATION

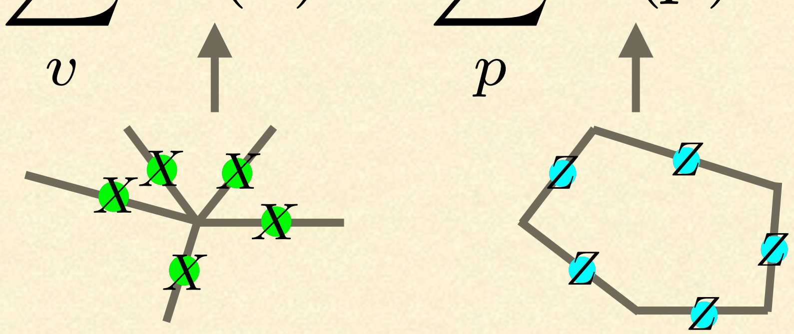
- Topological quantum codes - non-local DOFs, local generators.
- Toric code: high threshold, experimentally realizable (2 dim, 4-body terms, effective Hamiltonian of 2-body model).
- Color code: transversal implementation of logical gates, in particular $R_d = \mathrm{diag}(1, e^{2\pi i/2^d})$.
- Classification of quantum phases.
- Classification of systems with boundaries - beyond 2D.

TORIC CODE IN 2D



- qubits on edges
- X-vertex and Z-plaquette terms

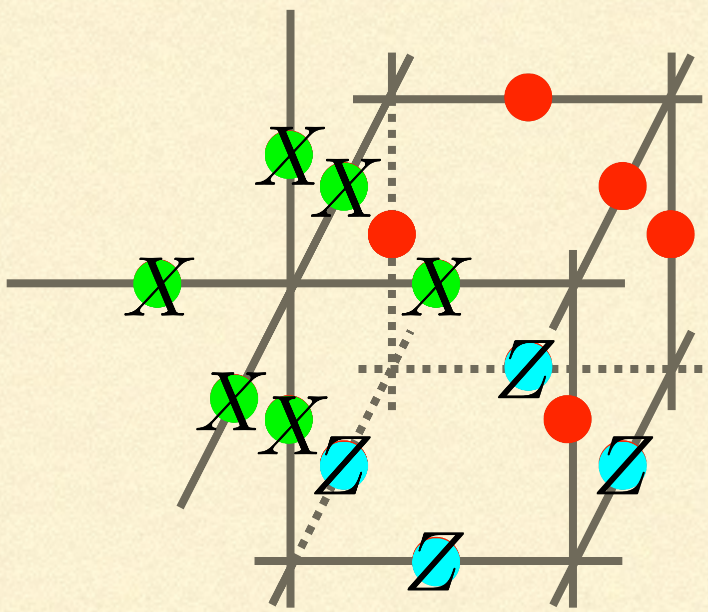
$$H = -\sum_v X(v) - \sum_p Z(p)$$



$$\forall v, p : \; [X(v), Z(p)] = 0$$

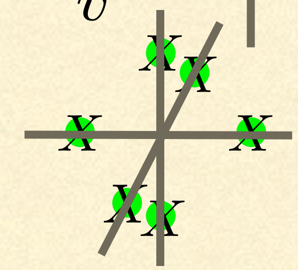
- code space C = ground states of H
- degeneracy(C) = 2^{2g}, where g - genus

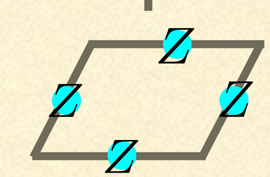
TORIC CODE IN 3D (OR MORE)



- qubits on edges
- X-vertex and Z-plaquette terms

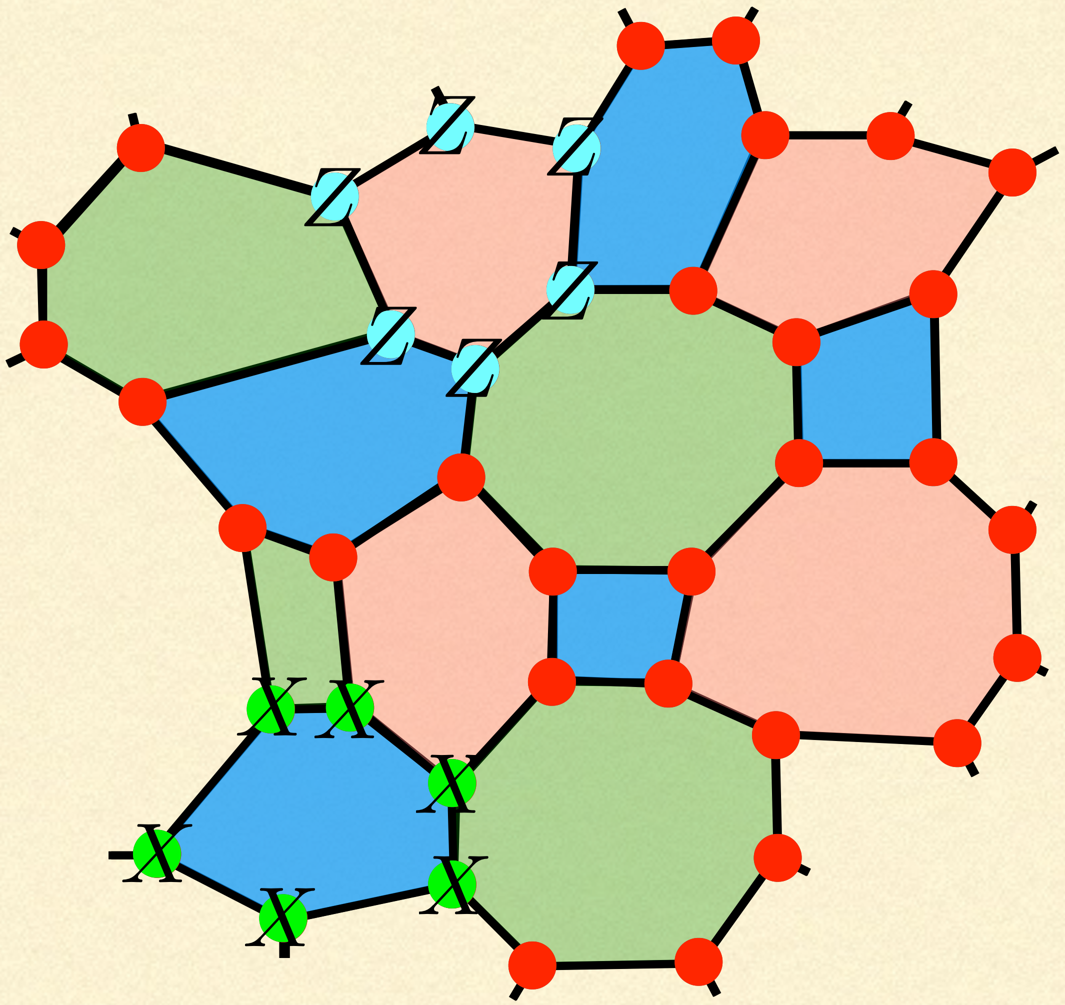
$$H = -\sum_v X(v) - \sum_p Z(p)$$





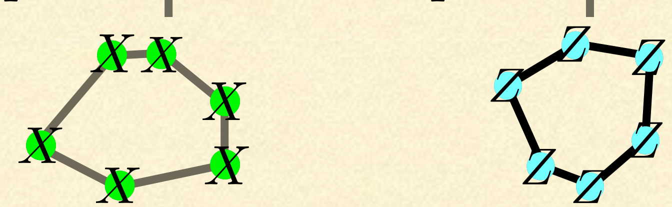
- lattice L in d dim - d-1 ways of defining toric code
- for 1< k <= d, $\mathbf{TC_k(L)}$: qubits - k-cells
X stabilizers - (k-1)-cells
Z stabilizers - (k+1)-cells

COLOR CODE IN 2D



- 2 dim lattice:
  - 3-valent
  - 3-colorable
- qubits on vertices
- plaquette terms

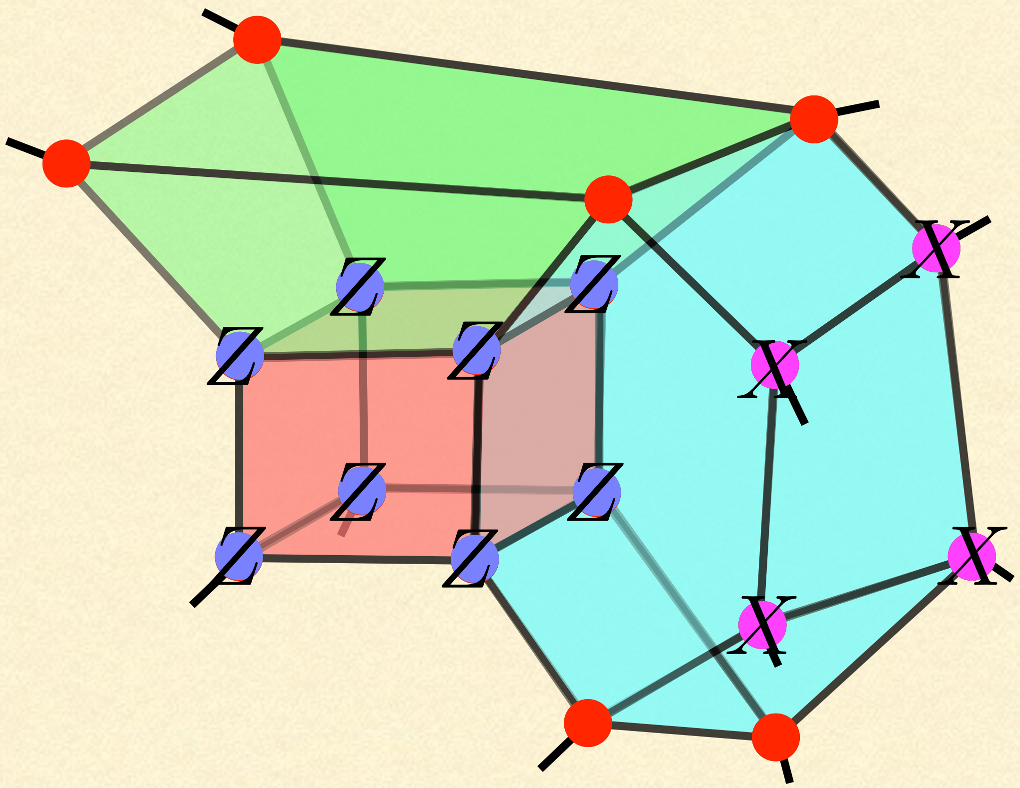
$$H = -\sum_p X(p) - \sum_p Z(p)$$



$$\forall p, p' : \ [X(p), Z(p')] = 0$$

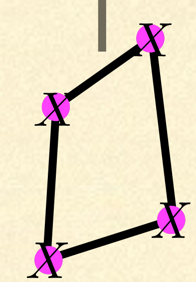
- code space C = ground states of H
- degeneracy(C) = 2^{4g}, where g - genus

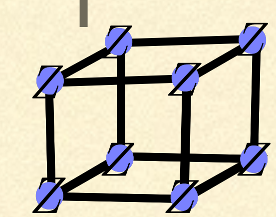
COLOR CODE IN 3D (OR MORE)



- d dim lattice:
  - (d+1)-valent
  - (d+1)-colorable
- qubits on vertices

$$H = -\sum_p X(p) - \sum_c Z(c)$$





- lattice L in d dim - d-1 ways of defining color code
- for 1< k <= d, **CC$_k$(L)**:

	qubits	- 0-cells
	X stabilizers	- (d+2-k)-cells
	Z stabilizers	- k-cells

WHY COLOR CODE?

- Transversally implementable gates: in 2 dim - Clifford group, in d dim - $R_d = \mathrm{diag}(1, e^{2\pi i/2^d})$, cf. **Bombin'13**.
- **Eastin & Knill'09**: for any nontrivial local-error-detecting quantum code, the set of logical unitary product operators is not universal.
- **Bravyi & König'13**: for a topological stabilizer code in d dim, a unitary implementable by a constant-depth quantum circuit and preserving the codespace implements an encoded gate from d$^{\text{th}}$ level of Clifford hierarchy.
- **Pastawski & Yoshida'14**: color code saturates many bounds!

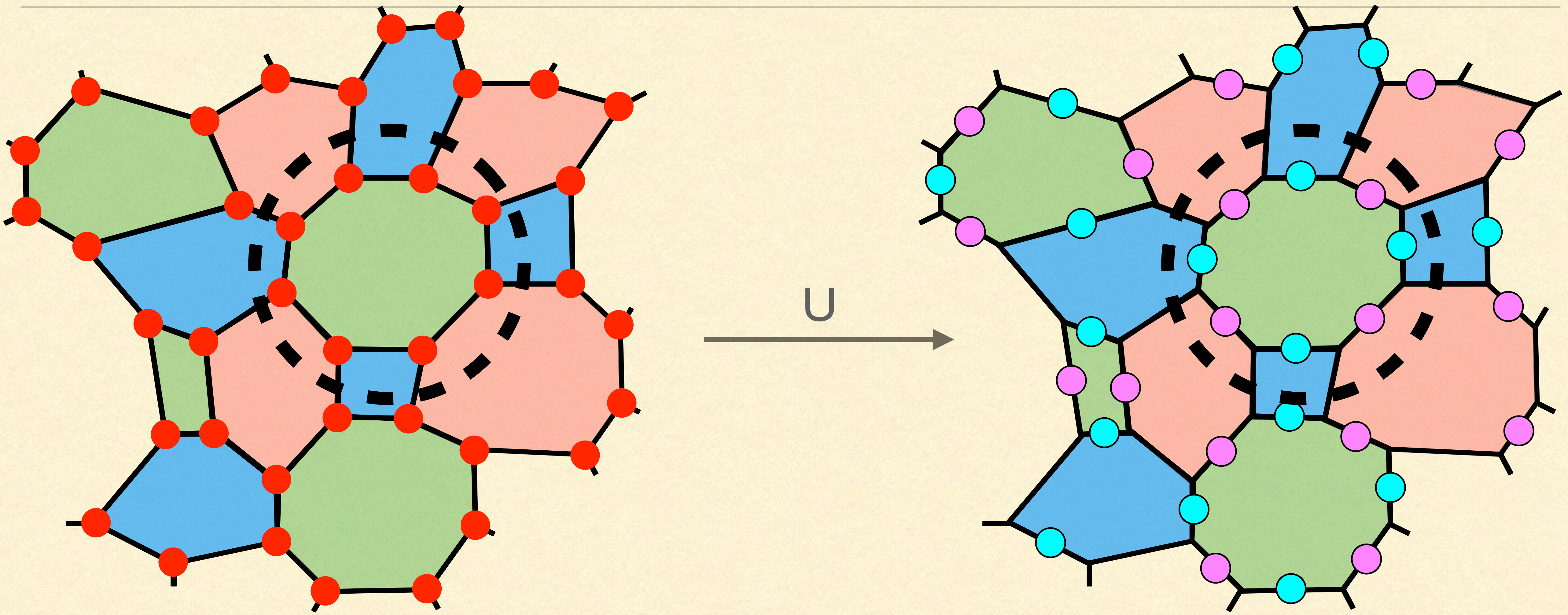
COLOR CODE VS TORIC CODE

- Color code and toric code - very similar but the same? Color code has transversal gates!
- **Chen et al.'10:** two gapped ground states belong to the same phase if and only if they are related by a local unitary evolution.
- **EQUIVALENCE** = up to adding/removing ancillas and local unitaries.
- **Yoshida'11, Bombin'11**: 2D stabilizer Hamiltonians with local interactions, translation and scale symmetries are equivalent to toric code*.
- What if no translation symmetry? TQFT argument!

OVERVIEW OF RESULTS

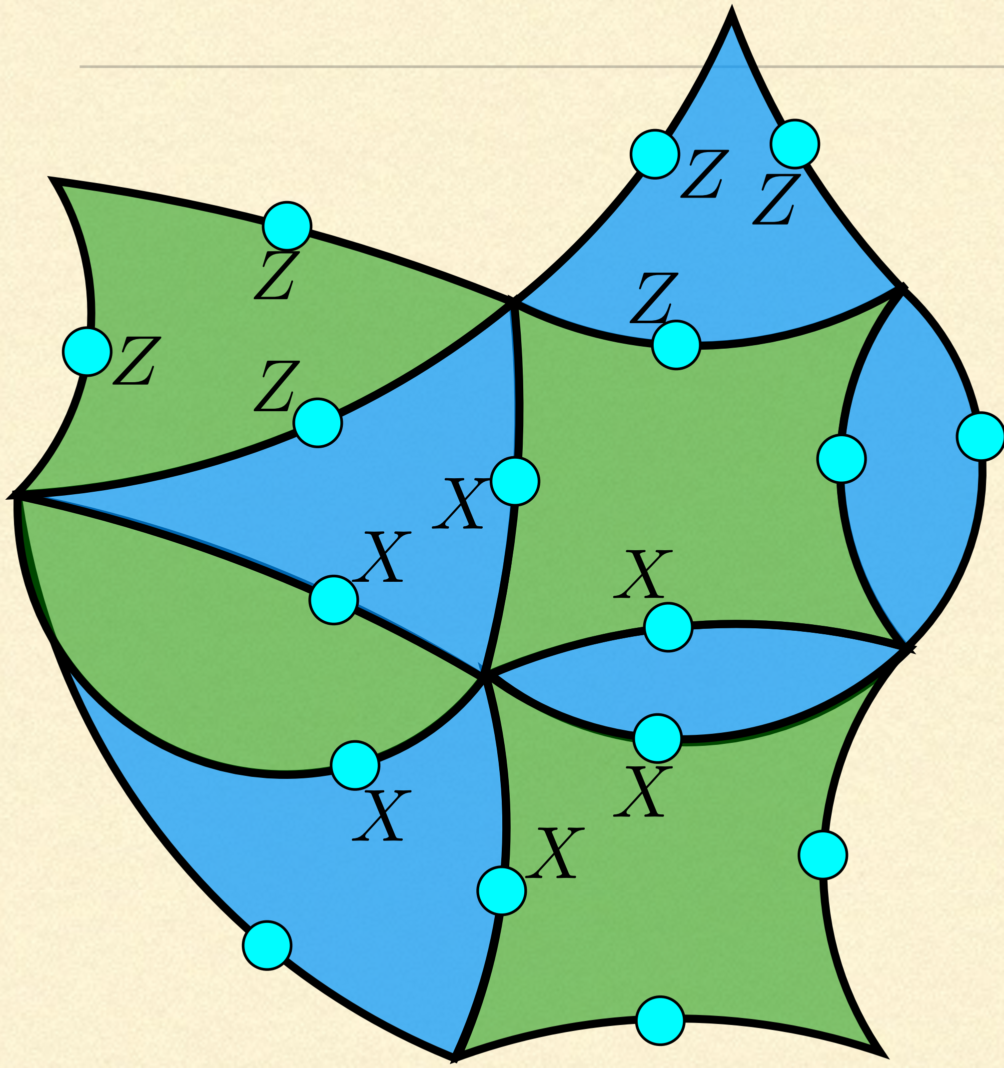
- **QUESTION:** how are color code and toric code related?
- **Result 1 (no boundaries):**
  color code = multiple decoupled copies of toric code.
- **Result 2 (boundaries):**
  color code = folded toric code.
- **Result 3 (logical gates):**
  non-Clifford gate $C^{d-1}Z$ in toric code.

TRANSFORMATION IN 2D

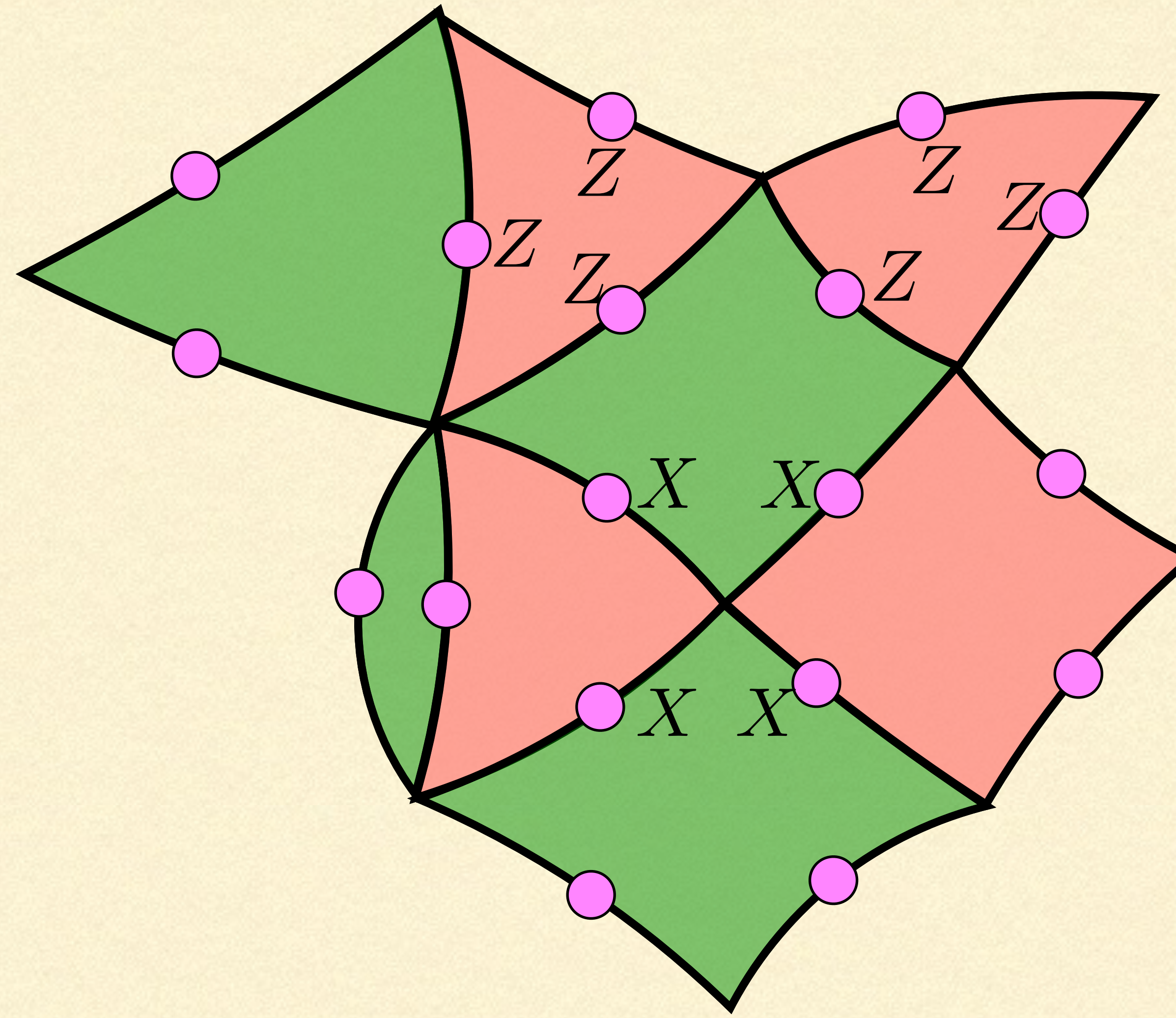


- Local unitary U between “red” and “turquoise/pink” qubits.
- Green plaquettes - local transformations.
- Every qubit belongs to exactly one green plaquette!

TRANSFORMATION IN 2D



shrink red plaquettes

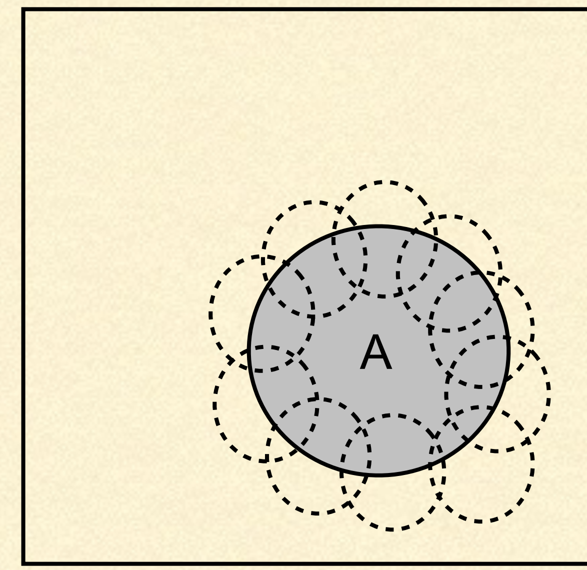


shrink blue plaquettes

- initial X/Z-plaquette terms transform into X-vertex/Z-plaquette terms!

EXISTENCE OF LOCAL UNITARY

- **Technical tool:** overlap group O, defined by restriction of stabilizer generators on A.

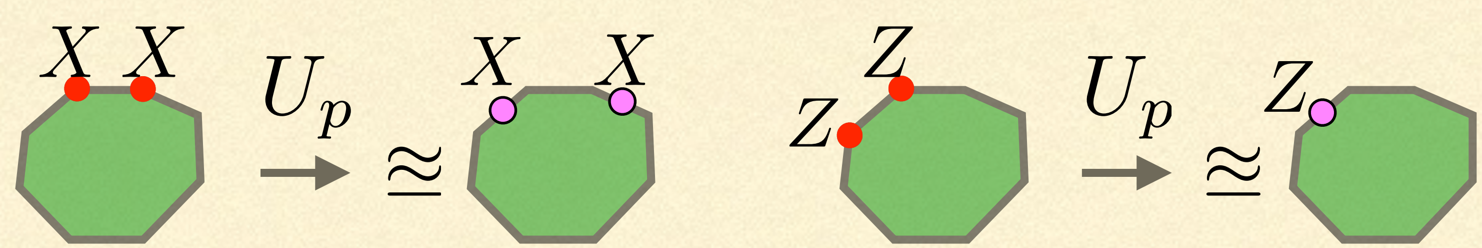


- Usually, O is non-Abelian and its canonical form

$$\mathcal{O} = \left\langle \begin{array}{cccccc} g_1, & \dots, & g_{n_1}, & g_{n_1+1} & \dots, & , g_{n_2} \\ g_{n_2+1}, & \dots, & g_{n_1+n_2} & & & \end{array} \right\rangle$$

- **Lemma:** if two overlap groups $\mathcal{O}_1 = \langle g_i \rangle$ and $\mathcal{O}_2 = \langle h_i \rangle$ have the same canonical form and $\{g_i\}$ and $\{h_i\}$ satisfy the same (anti)commutation relations, then there exists a Clifford unitary U, such that $\forall i : h_i = U g_i U^\dagger$.

- Color code in 2 dim:



EQUIVALENCE IN D DIMENSIONS

Theorem: there exists a unitary $U = \bigotimes_\delta U_\delta$, which is a tensor product of local terms with disjoint support, such that U transforms the color code into $n = \binom{d}{k-1}$ decoupled copies of the toric code.

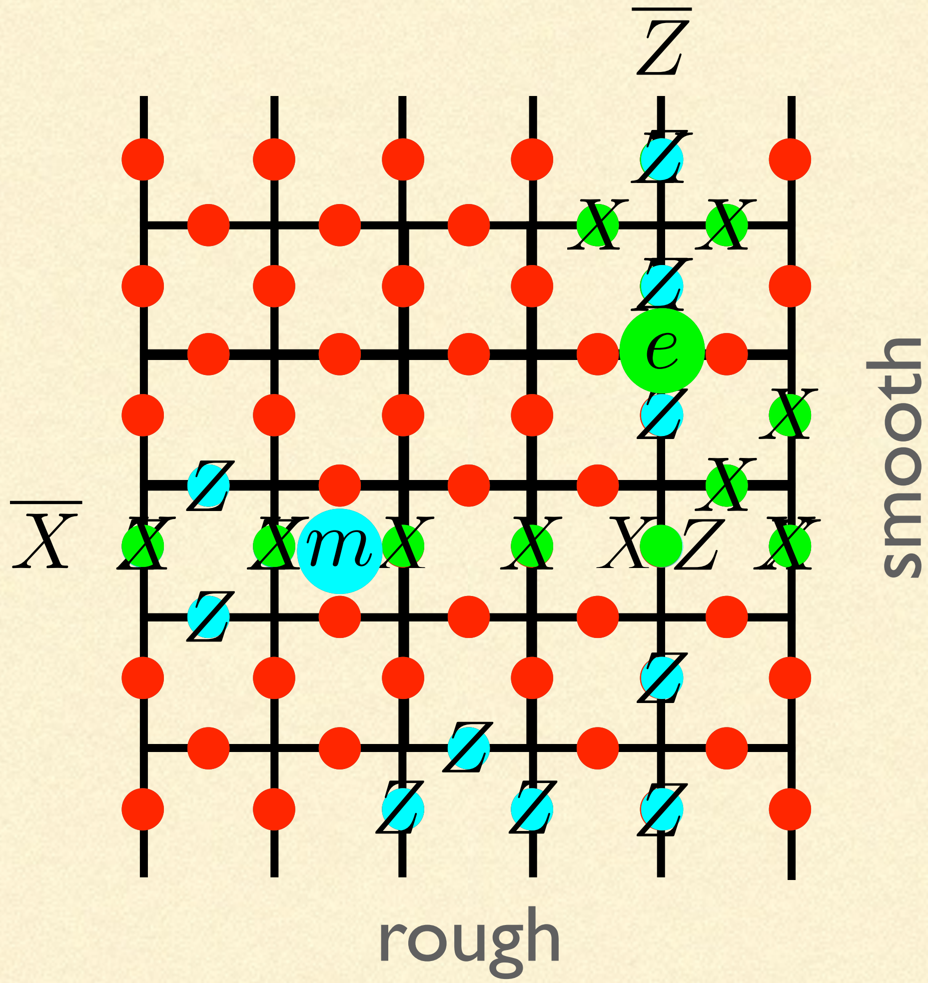
$$U\left[CC_k(\mathcal{L})\right]U^\dagger = \bigotimes_{i=1}^{n} TC_{k-1}(\mathcal{L}_i)$$

obtained from $\mathcal{L}$ by
local deformations

- **CC$_k$(L)**: qubits on 0-cells, X- and Z-stabilizers on (d+2-k)-cells and k-cells
  TC$_k$(L): qubits on k-cells, X- and Z stabilizers on (k-1)-cells and (k+1)-cells

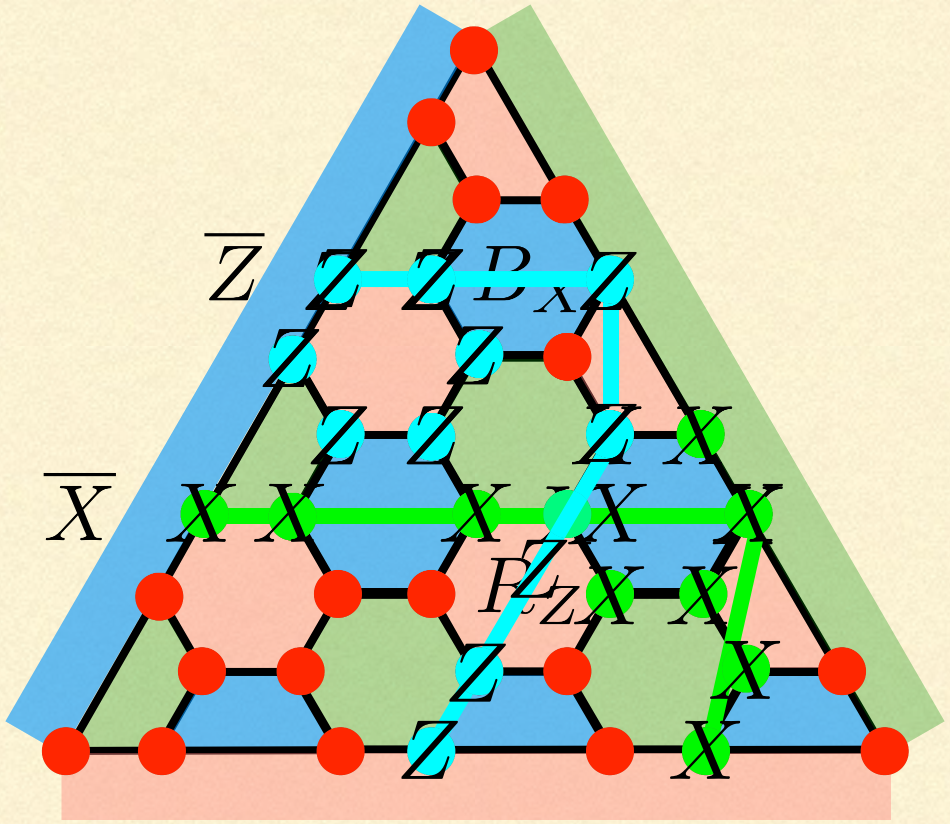
CODES WITH BOUNDARIES

- 2 dim toric code with boundaries: rough and smooth
- excitations: electric and magnetic, e, m, and composite, $\epsilon = e \times m$
- 1 logical qubit



CODES WITH BOUNDARIES

- 2 dim color code with boundaries: red, green and blue
- excitations: red/green/blue of X and Z-type, $R_X, R_Z, G_X, G_Z, B_X, B_Z$
- 1 logical qubit

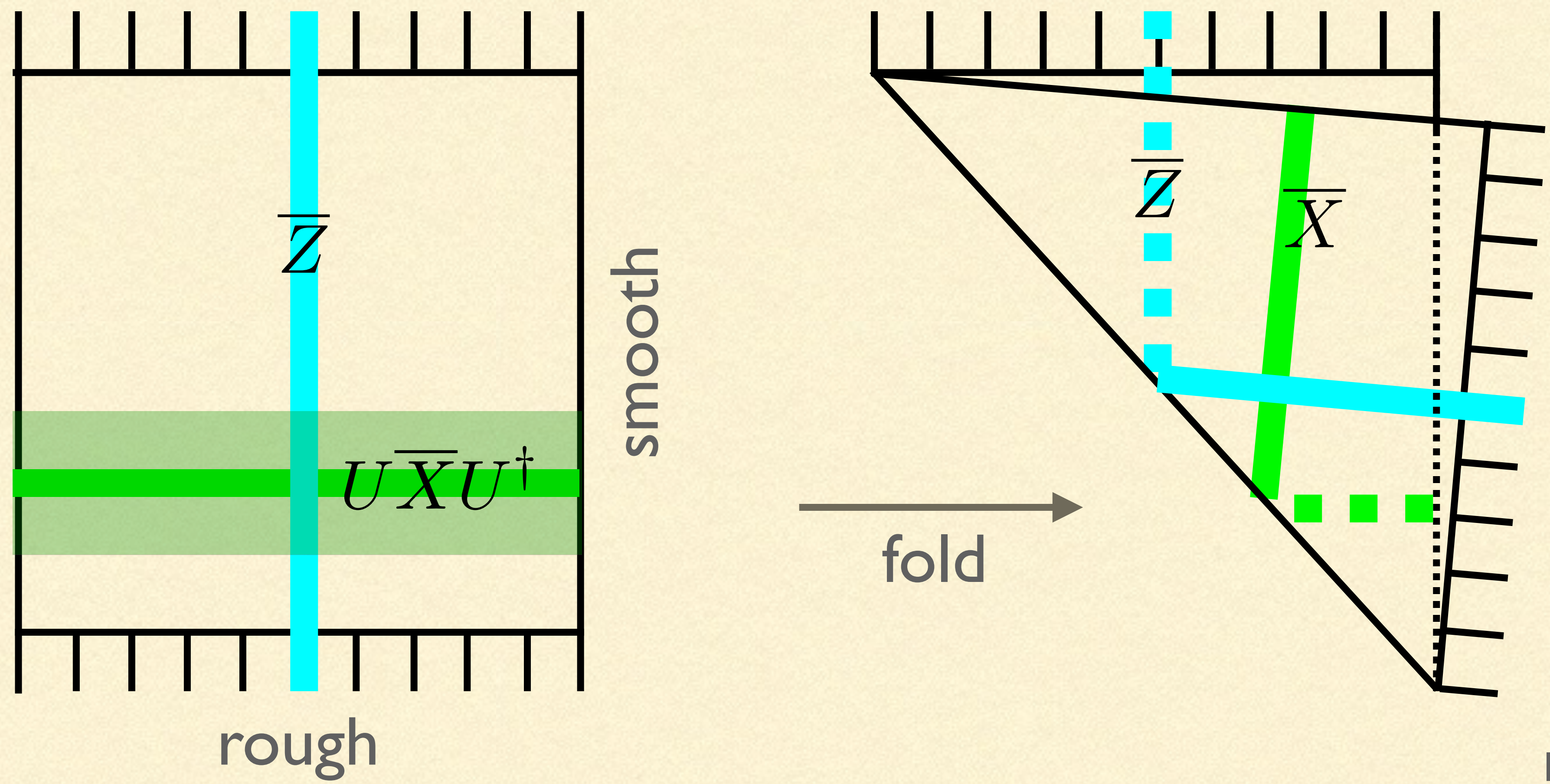


NECESSITY OF FOLDING

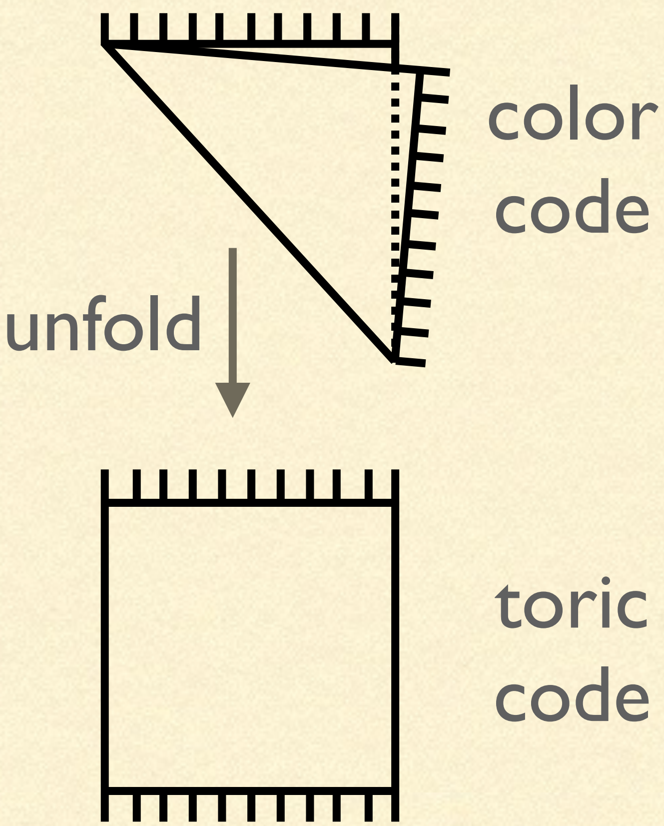
- We want to relate color code and toric code. But color code has transversal Hadamard gate!
- If U - local unitary implementing logical Hadamard in toric code, then

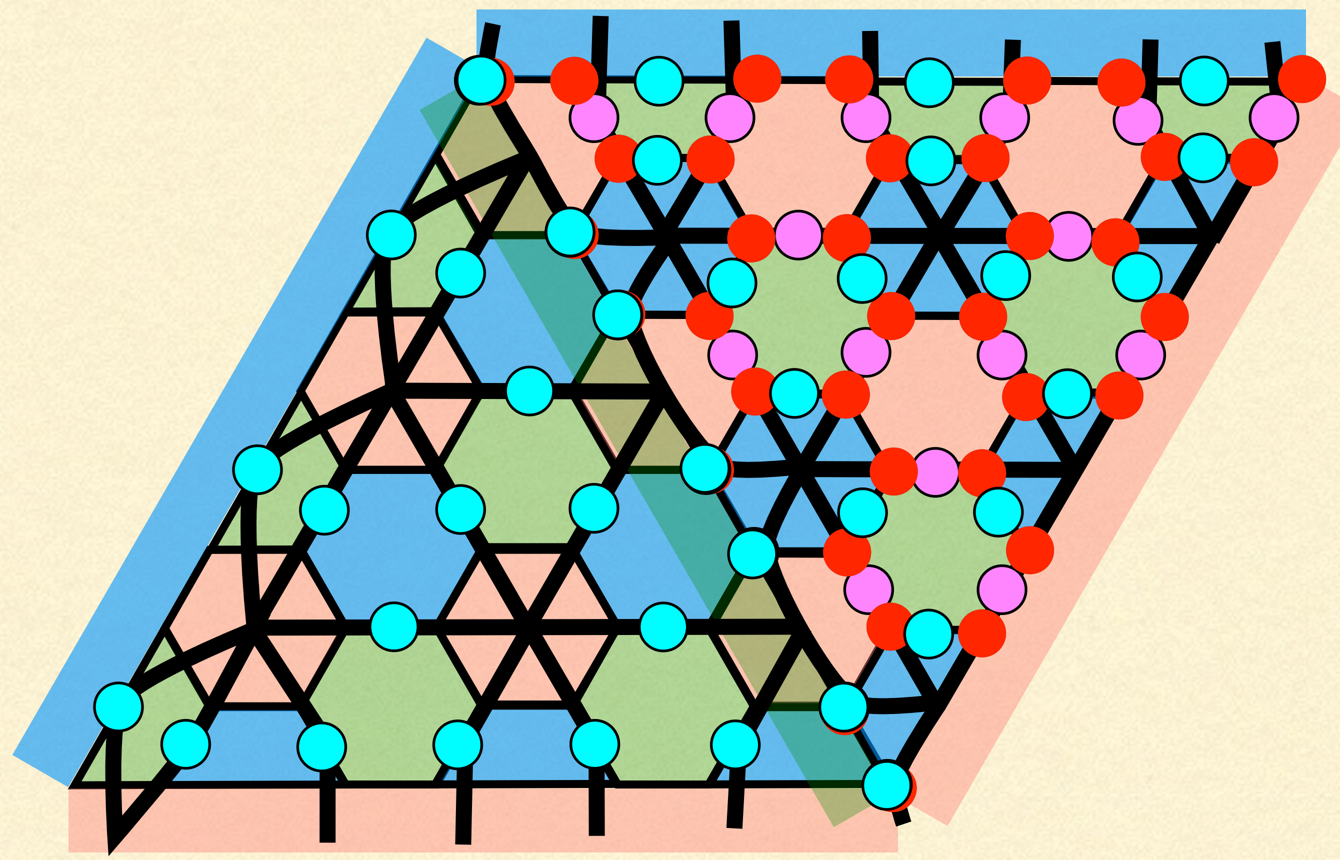
$$X \overset{H}{\longleftrightarrow} Z$$

$$\overline{X} \overset{U}{\longleftrightarrow} \overline{Z}$$



COLOR CODE UNFOLDED





- local unitaries on green plaquettes and red/blue plaquettes along the green boundary
- **Theorem**: color code in d dim with d+1 boundaries is equivalent to multiple copies of toric code attached along (d-1)-dimensional boundary.

ANYONS AND CONDENSATION

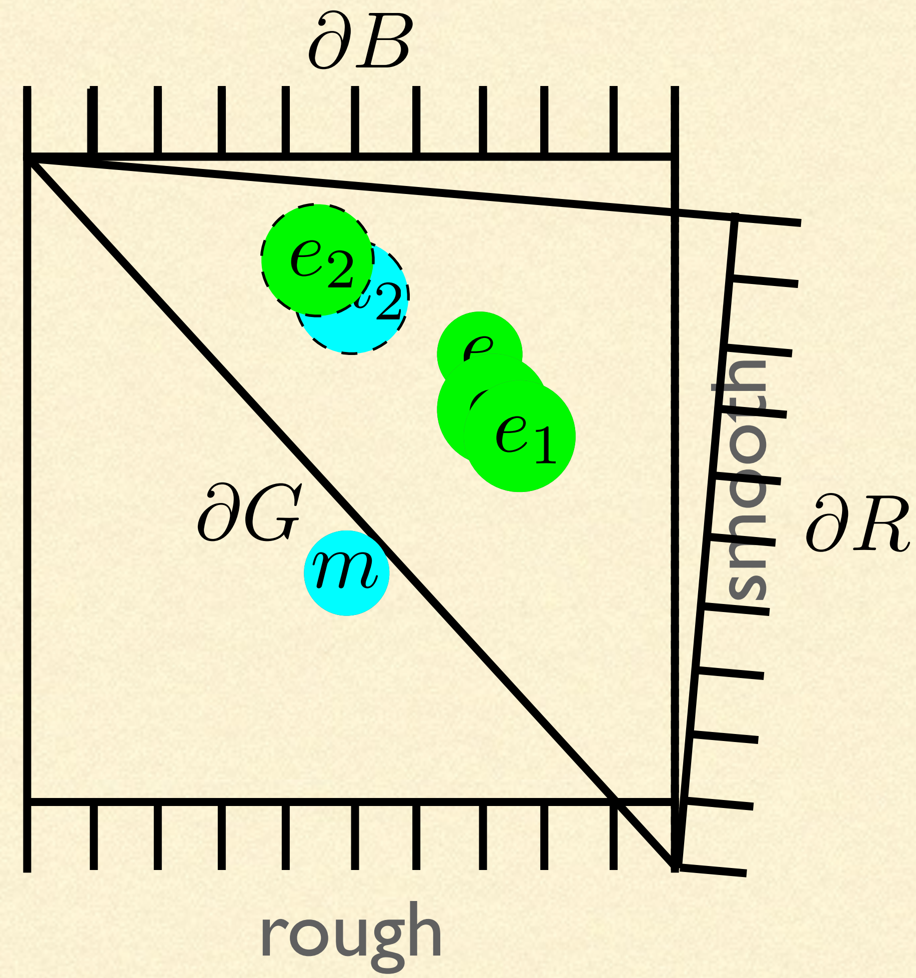
- **Fact**: anyons condensing into a gapped boundary have mutually trivial statistics.
- **Toric code:** e - rough, m - smooth.
- **Folded toric code:**

$$\partial R = \{1, e_1, m_2, e_1 m_2\}$$
$$\partial B = \{1, e_2, m_1, e_2 m_1\}$$
$$\partial G = \{1, e_1 e_2, m_1 m_2, \epsilon_1 \epsilon_2\}$$

- Correspondence between anyonic excitations in toric code and color code:

$e_1 \equiv R_X \qquad e_2 \equiv B_X$

$m_2 \equiv R_Z \qquad m_1 \equiv B_Z$



TRANSVERSAL GATES

- Gates in d^{th} level of Clifford hierarchy
  $R_d|x\rangle = e^{2\pi i x/2^d}|x\rangle \qquad C^{d-1}Z|x_1,\ldots,x_d\rangle = (-1)^{x_1\ldots x_d}|x_1,\ldots,x_d\rangle$

- Color code in d dim has transversally implementable logical R_d.

- Start with d copies of toric code, switch to color code by local unitary, apply logical R_d, switch back to toric code = implements logical $C^{d-1}Z$ on d copies of toric code in d dim.

- Toric code saturates Bravyi-König classification!

SUMMARY

- **No boundaries:** color code = multiple decoupled copies of toric code.
- **Boundaries:** color code = folded toric code.
- **Non-Clifford gate $C^{d-1}Z$** in d copies of toric code implementable via a local unitary transformation.
- Reverse the procedure: start with multiple copies and apply local transformations to obtain new codes, cf. Brell'14: G-color codes.
- Insights into classification of TQFTs with boundaries in 2 dim or more.

TRANSFORMATION IN 2D