Parafermion stabilizer codes

- Alexey A. Kovalev

Phys. Rev. A 90, 042326 (2014)

Collaborators:
Utkan Güngördü
Rabindra Nepal
Ilya Dumer
Leonid Pryadko

Outline

- Motivation to consider parafermion codes
- From qudit codes to parafermion codes and Jordan-Wigner transformation
- Mapping from qudits to parafermion modes and from parafermion modes to qudits
- Parafermion toric code with adjustable protection against Z_D-charge (parity) non-conserving errors

Majorana modes

$$\begin{cases} \gamma_j^A = c_j + c_j^\dagger \\ \gamma_j^B = i(c_j - c_j^\dagger) \end{cases} \Rightarrow \begin{array}{c} (\gamma_j^\alpha)^\dagger = (\gamma_j^\alpha) \\ \left\{\gamma_i^\alpha, \gamma_j^\beta\right\} = 2\delta_{ij}\delta_{\alpha\beta} \end{array}$$

They are real and imaginary parts of a creation operator.

Can be realized in systems with interactions

$$H = \sum_j -t(c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j) - \Delta(c_j c_{j+1} + c_{j+1}^\dagger c_j^\dagger) - \mu c_j^\dagger c_j$$

A. Yu. Kitaev (2001)

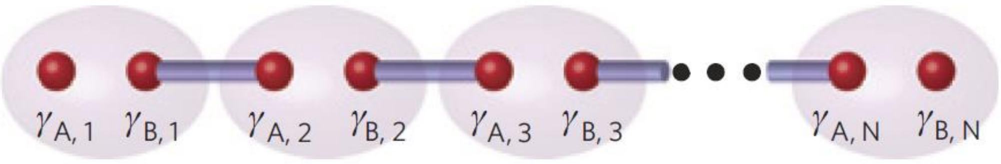
$$\mu = 0,\ t = |\Delta|$$

$$H = -it \sum_j^{N-1} \gamma_{B,j}\gamma_{A,j+1}$$

Fermionic quantum computation, Annals of Physics, Vol. 298, Iss. 1 (2002) pp.210-226

Majorana fermion codes, Bravyi, Terhal, Leemhuis, New J. Phys. **12**, 083039 (2010)

$\gamma_{A,1},\ \gamma_{B,N}$ -- drop out from Hamiltonian and allow us to form an artificial fermion.

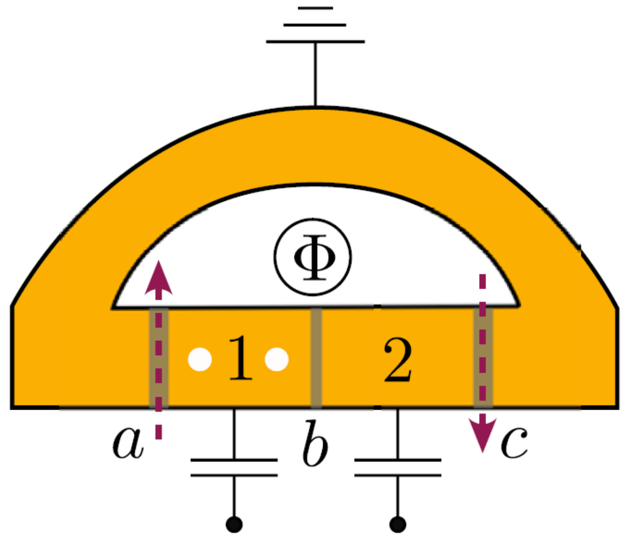


At low energies, the whole wire behaves as one fermion.

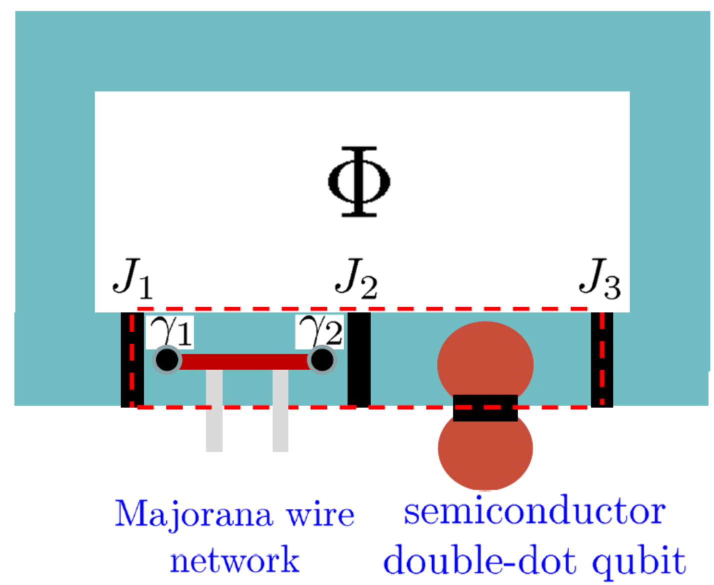
J. Alicea, Y. Oreg, G. Refael, F. von Oppen & M. P. A. Fisher Nature Physics 7, 412–417 (2011)

Measurement and coupling via Aharonov-Casher effect

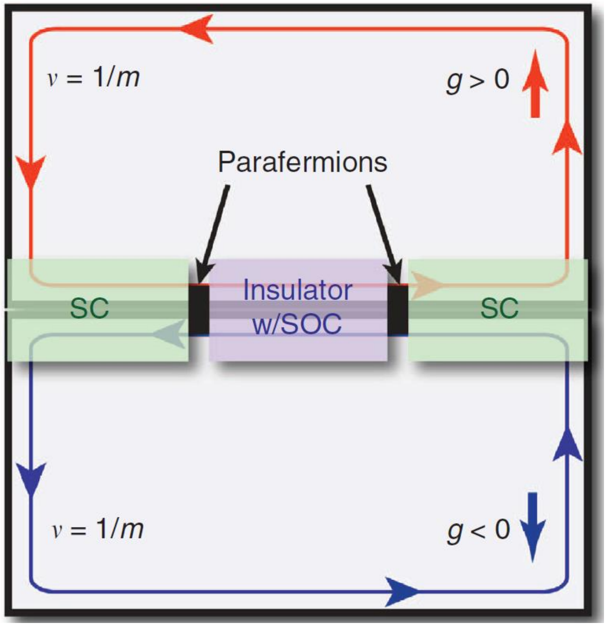
Hassler, Akhmerov, Hou, Beenakker,
NJP 12, 125002 (2010)



Bonderson, Lutchyn,
Phys. Rev. Lett. 106, 130505 (2011)
Jiang, Kane, Preskill
Phys. Rev. Lett. 106, 130504 (2011)



Networks of parafermion wires



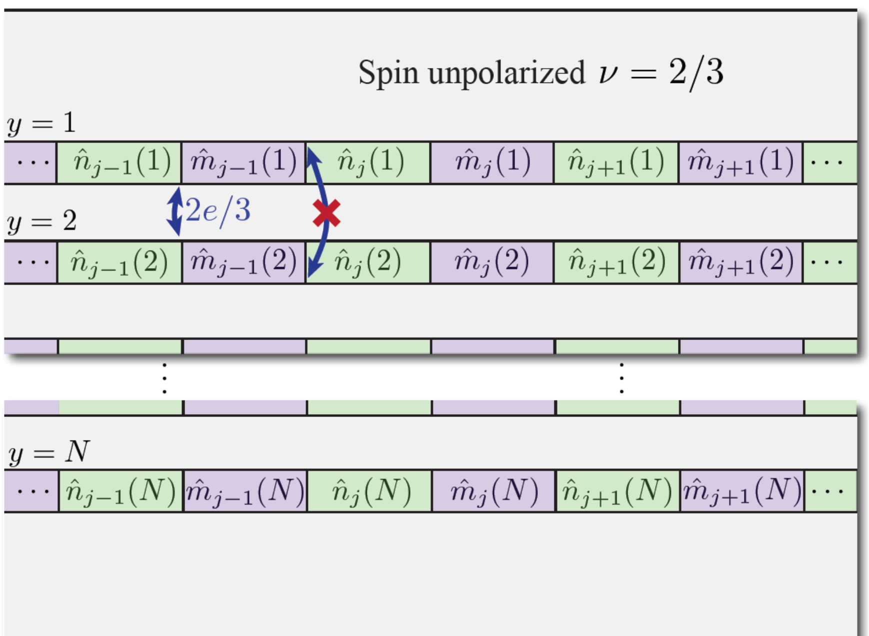
Braiding properties:

$$\gamma_1 \to e^{ik2\pi/D}\gamma_2 \qquad \text{vs} \qquad \gamma_1 \to \gamma_2$$
$$\gamma_2 \to e^{i(1-k)2\pi/D}\gamma_1^\dagger\gamma_2^2 \qquad \gamma_2 \to -\gamma_1^2$$

parafermions majoranas

Clarke, Alicea, Shtengel, Nature Commun. 4, 1348 (2013)

Networks of parafermion wires can be used to obtain Fibonachi anyons.



Mong, Clarke, Alicea, Lindner, Fendley, Nayak, Oreg, Stern, Berg, Shtengel, Fisher, Phys. Rev. X 4, 011036 (2014)

Qudit stabilizer codes

Let us consider generalized Pauli group: $\mathscr{P}_n = \omega^m \{I, X, Y, Z\}^{\otimes n},\ m = 0, ..., D-1$

$$\omega = e^{2\pi i/D}$$

$$X^D = Z^D = \mathbb{1}, \qquad ZX = \omega XZ$$

$$X = \sum_{j=0}^{D-1} |j+1\rangle \langle j|\,, \qquad Z = \sum_{j=0}^{D-1} \omega^j |j\rangle \langle j|$$

An $[[n,k,d]]$ stabilizer code $\mathcal{Q}$ is a D^k-dimensional subspace of the Hilbert space $\mathcal{H}_D^{\otimes n}$ stabilized by an Abelian stabilizer group $\mathscr{S} = \langle G_1, ..., G_{n-k} \rangle$, $\mathbb{1}\omega^j \notin \mathscr{S}$; $\mathcal{Q} = \{|\psi\rangle : S|\psi\rangle = |\psi\rangle, \forall S \in \mathscr{S}\}$.

Since the code is stabilized by the stabilizer group (syndrome measurements) we actually measure errors but not the stored information.

Since errors are measured by Pauli operators, any non-Pauli error is projected to a Pauli one -> <u>It is sufficient to treat only Pauli errors</u>.

E. M. Rains, IEEE Trans. Inf. Theor. 45, 1827 (1999); A. Ashikhmin and E. Knill, IEEE Trans. Inf. Theor. 47, 3065 (2001); D. Schlingemann and R. F. Werner, Phys. Rev. A 65, 012308 (2001).

Qudit codes: matrix representation

Pauli operators are mapped to two strings, $\mathbf{v}, \mathbf{u} \in \{0, D-1\}^n$, $U \equiv \omega^{m'} X^{\mathbf{v}} Z^{\mathbf{u}} \rightarrow (\mathbf{v}, \mathbf{u})$, where $X^{\mathbf{v}} = X_1^{v_1} X_2^{v_2} ... X_n^{v_n}$ and $Z^{\mathbf{u}} = Z_1^{u_1} Z_2^{u_2} ... Z_n^{u_n}$. A product of two quantum operators corresponds to a sum $(\mathrm{mod}\, D)$ of the corresponding pairs $(\mathbf{v}_i, \mathbf{u}_i)$.

In this representation, a stabilizer code is represented by parity check matrix written in binary form for X and Z Pauli operators so that, e.g. XIYZYI=-(XIXIXI)x(IIZZZI) -> (101010)|(001110).

$$\begin{array}{cc} & \qquad \text{Ax} \qquad\qquad\qquad\qquad \text{Az} \\ \boldsymbol{H=} & \left(\begin{array}{ccccc|ccccc} 1 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 1 & 1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & -1 & 0 & -1 & 0 & 0 \end{array}\right) \end{array}$$

Example of a parity check matrix *H* of [[5,1,3]] code written in X-Z form.

$$A_X A_Z^T = A_Z A_X^T (\mathrm{mod} D)$$

Necessary and sufficient condition for existence of stabilizer code with stabilizer commuting operators corresponding to ***H***.

$$(z|x) \odot (z'|x')^T = zx'^T - xz'^T$$

Row orthogonality with respect to symplectic product.

Parity check matrix for a Calderbank-Shor-Steane (CSS) code: $H = \left(\begin{array}{c|c} G_X & 0 \\ 0 & G_Z \end{array}\right)$, $G_X G_Z^T = 0$ ⟸ commutativity

Qudit codes: error correction

1. Measure stabilizer generators to obtain syndrome of error *E* $\Leftrightarrow$ $H \odot E^T$

2. Correct error according to syndrome.

- The correctable error set ***Ec*** is defined by:

  If E_1 and E_2 are in *Ec*, then one of the two conditions hold:

  1. $E_2^\dagger E_1 \notin H_\perp \setminus \mathscr{S}$ distinct error syndromes

  2. $E_2^\dagger E_1 \in \mathscr{S}$ degenerate code

- The detectable error set ***Ed*** is defined by:

  If *E* is in *Ed*, then one of the two conditions hold:

  1. $E \notin H_\perp \setminus \mathscr{S}$ distinct error syndromes

  2. $E \in \mathscr{S}$ degenerate code

Syndrome of (*I I I Y I*) error:
$$\left(\begin{array}{ccccc|ccccc} 1 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 1 & 1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & -1 & 0 & -1 & 0 & 0 \end{array}\right) \odot (\overset{E_X}{00010} | \overset{E_Z}{00010})^T = \begin{pmatrix} 0 \\ -1 \\ -1 \\ -1 \end{pmatrix}$$

The distance of a quantum stabilizer code is defined as the minimal weight of all undetectable errors, i.e. Hamming weight of $E_X \vee E_Z$

Jordan-Wigner transformation

$$X^D = Z^D = \mathbb{1}, \qquad ZX = \omega XZ$$

$$X = \sum_{j=0}^{D-1} |j+1\rangle \langle j| , \qquad Z = \sum_{j=0}^{D-1} \omega^j |j\rangle \langle j|$$

Tensor products of qudit Pauli operators can be mapped to tensor products of parafermion operators by employing the Jordan-Wigner transformation.

$$\boxed{\begin{aligned} \gamma_{2j-1} &= \left(\prod_{k=1}^{j-1} X_k\right) Z_j , \\ \gamma_{2j} &= \omega^{(D-1)/2} \left(\prod_{k=1}^{j-1} X_k\right) Z_j X_j \end{aligned}}$$

Commutativity relations imply non-local character of parafermion operators.

$$\gamma_j^D = \mathbb{1}, \quad \gamma_j \gamma_k = \omega \gamma_k \gamma_j \quad (j < k, \quad \omega = e^{i2\pi/D})$$

From quantum clock model to parafermion codes

$$H_3 = -J \sum_{j=1}^{n-1} (Z_j^\dagger Z_{j+1} + Z_{j+1}^\dagger Z_j)$$

Three state clock model with h=0

Fendley, arXiv:1209.0472

Apply Jordan-Wigner transformation:

$$H = iJ \sum_{j=1}^{n-1} (\gamma_{2j}^\dagger \gamma_{2j+1} - \gamma_{2j+1}^\dagger \gamma_{2j})$$

This Hamiltonian corresponds to the sum of commuting operators.
The code space is stabilized by the Abelian group generated from this set.

$$\boxed{\langle i\gamma_2^\dagger \gamma_3, i\gamma_4^\dagger \gamma_5, \ldots\ldots, i\gamma_{2n-2}^\dagger \gamma_{2n-1} \rangle}$$

In the absence of parity breaking interactions this code protects against local errors.

Logical operators can be identified as

$$\bar{Z} = \gamma_1$$
$$\bar{X} = \gamma_{2n}$$

Parafermion stabilizer codes

We consider tensor products of parafermion operators corresponding to *2n* modes and denote this group by $\mathrm{PF}(D, 2n)$

Parafermion stabilizer codes $C_{\mathcal{S}_{PF}}$, similar to qudit stabilizer codes, are completely determined by their corresponding stabilizer group, which in our case is $\mathcal{S}_{PF} \subseteq \mathrm{PF}(D, 2n)$. We list the defining properties of parafermion stabilizer codes as:

- Elements of $\mathcal{S}_{PF}$ are parity-preserving operators.
- $\mathcal{S}_{PF}$ is an Abelian group not containing $\omega^j \mathbb{1}$ where $j \in \mathbb{Z}_D$ and $j \neq 0$.
- $[[n, k, d]]$ code is a D^k dimensional subspace stabilized by the stabilizer group.

The parity condition can be also written as commutativity with the charge operator:

$$Q = \prod_{j=1}^{n} \gamma_{2j-1}^{\dagger} \gamma_{2j}$$

The code protects against low weight errors (low weight tensor products of parafermions)!

Parafermion codes: matrix representation

Arbitrary elements of $\mathrm{PF}(D, 2n)$ can be written as $\omega^{\lambda}\gamma^{\boldsymbol{\alpha}}$ where $\lambda \in \mathbb{Z}_D$ and $\gamma = \gamma_1^{\alpha_1} \ldots \gamma_{2n}^{\alpha_{2n}}$ with $\alpha = (\alpha_1, \ldots, \alpha_{2n}) \in \mathbb{Z}_D^{2n}$ and by convention the terms are arranged in increasing order in their indices.

We use a matrix representation of $\mathcal{S}_{PF} = \langle S_1, \ldots, S_l \rangle = \langle \gamma^{\boldsymbol{\alpha}_1}, \ldots, \gamma^{\boldsymbol{\alpha}_l} \rangle$ whose rows are given by $\boldsymbol{\alpha}_i$, that is

$$\boxed{S_{PF} = \begin{pmatrix} \boldsymbol{\alpha}_1 \\ \vdots \\ \boldsymbol{\alpha}_l \end{pmatrix}}$$

$$S_{PF} = \begin{pmatrix} -1 & 1 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 & 0 & -1 & 0 & 1 \end{pmatrix}$$

Example of a parity check matrix of [[8,1,3]] parafermion code for D=3.

$$\boxed{S_{PF} \Lambda S_{PF}^T = 0 \mod D}$$

Necessary and sufficient condition for existence of stabilizer code with stabilizer commuting operators corresponding to S_{PF}

Where $\Lambda = \begin{pmatrix} 0 & 1 & 1 & \ldots & 1 \\ -1 & 0 & 1 & \ldots & 1 \\ -1 & -1 & 0 & \ldots & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ -1 & -1 & -1 & \ldots & 0 \end{pmatrix}$

For [[n,k,d]] code $k = n - \mathrm{rank} S_{PF}$

$\mathcal{L}(\mathcal{S}_{PF}) = \mathcal{C}(\mathcal{S}_{PF}) \setminus \mathcal{S}_{PF}$ $\quad d = \min_{\gamma^{\boldsymbol{\alpha}} \in \mathcal{L}(\mathcal{S}_{PF})} |\gamma^{\boldsymbol{\alpha}}|$

$\boxed{L_{PF} \text{ is the matrix form of } \mathcal{L}(\mathcal{S}_{PF})}$

Mapping from parafermion to qudit code

Lemma *Any parafermion stabilizer code with parameters* $[[2n, k, d]]_D$ *and stabilizer group* $\mathcal{S}_{PF}$ *can generate a* $[[2n, 2k, d']]_D$ *qudit CSS code.*

Proof. Consider the check matrix

$$S_{CSS} = \begin{pmatrix} S_{PF}\Lambda & 0 \\ 0 & S_{PF} \end{pmatrix}$$

For a parafermion code, $k = n - \mathrm{rank}(S_{PF})$ whereas for the CSS code $k' = 2n - 2 \times \mathrm{rank}(S_{PF}) = 2k$ (Λ is full-rank matrix). Hence S_{CSS} is the check matrix of a $[[2n, 2k, d']]$ CSS code.

The corresponding logical operator matrices L_{PF} and $L_{PF}\Lambda$, behave like X- and Z-type logical qudit operators.

For $D = 2$ this procedure maps a Majorana fermion code to weakly self-dual CSS code. Unfortunately, for $D > 2$ this mapping becomes non-local, i.e., a local qudit operator will generally map to a non-local parafermion operator.

For *D=2* this mapping is given in Bravyi, Terhal, Leemhuis, New J. Phys. **12**, 083039 (2010)

Mapping from qudit to parafermion code

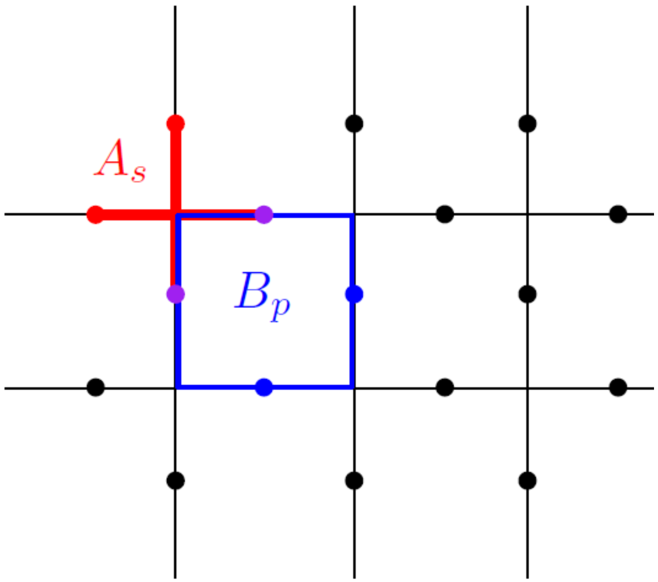
Lemma *Every $[[n,k,d]]_D$ stabilizer code can be mapped onto a $[[4n,k,2d]]_D$ parafermion stabilizer code, encoding 4 parafermion modes into a single qudit.*

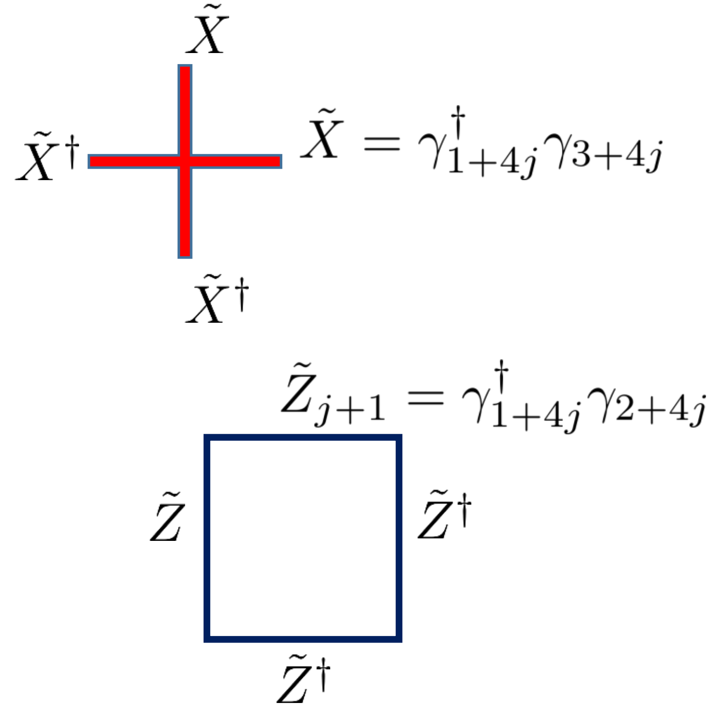
Proof. Let us define the operators

$$\boxed{\begin{aligned} \tilde{Z}_{j+1} &= \gamma^{\dagger}_{1+4j}\gamma_{2+4j}, \qquad \tilde{X}_{j+1} = \gamma^{\dagger}_{1+4j}\gamma_{3+4j} \\ \tilde{Q}_{j+1} &= \gamma^{\dagger}_{1+4j}\gamma_{2+4j}\gamma^{\dagger}_{3+4j}\gamma_{4+4j} \end{aligned}}$$

It is straightforward to show that $\langle \tilde{X}_j, \tilde{Z}_j \rangle$ generate the embedded Weyl group $\mathcal{W}_D^{\otimes k} \subseteq \mathrm{PF}(D,2n)$ (that is, $\tilde{Z}_i\tilde{X}_j = \omega \tilde{X}_j \tilde{Z}_i \delta_{ij}$ and $\tilde{X}_j^D = \tilde{Z}_j^D = \mathbb{1}$) and are parity-preserving. We can treat $\mathcal{L}(\mathcal{S}_{PF}) = \langle \tilde{X}_j, \tilde{Z}_j \rangle$ as the logical operators of a stabilizer group $\mathcal{S}_{PF} = \langle \bar{Q}_j \rangle$. This makes the purpose of the additional fourth mode (which does not appear in the logical operators) clear: without it, the stabilizer group would include a non-parity-preserving operator. Finally, since every Weyl operator is mapped to a parafermion operator with two modes, the distance of the new code is $2d$.

Parafermion toric code





Parafermion code is constructed by designating 4 parafermion modes to each qudit of original code.

Z_D charge conservation and error model

- Z_D charge (parity) breaking errors are less likely to occur (ideally do not occur at all).
- Thus it makes sense to define code distance with respect to errors that conserve Z_D charge.
- This is only relevant to codes containing logical operators not conserving Z_D charge.
- In addition, one can define distance with respect to errors not conserving Z_D charge.
- The mapping form qudit codes to parafermion codes presented earlier will only result in codes with no logical operators violating Z_D charge.
- Thus it makes sense to construct parafermion codes directly without employing the mapping
- For local errors one can also define the radius of logical operators preserving Z_D charge.

$$l_{\text{con}} = \min_{\substack{\gamma \in \mathcal{L}(\mathcal{S}_{PF}) \\ \sum_i \alpha_i = 0 \mod D}} \operatorname{diam}[\operatorname{Supp}(\gamma^{\alpha})]$$

- Generalization of Kitaev's model leads to $l_{\text{con}} = 2n$

$$H = iJ \sum_{j=1}^{n-1} (\gamma_{2j}^{\dagger}\gamma_{2j+1} - \gamma_{2j+1}^{\dagger}\gamma_{2j})$$

Non-prime D case

Theorem *Let $\mathcal{S}_{PF}$ be a parafermion stabilizer code in $PF(D,2n)$ where D is allowed to be composite, let $|\mathcal{S}_{PF}|$ denote the order of $\mathcal{S}_{PF}$ and let $|C_{\mathcal{S}_{PF}}|$ be the dimension of codespace. Then the following equation holds:*

$$|C_{\mathcal{S}_{PF}}||\mathcal{S}_{PF}| = D^n.$$

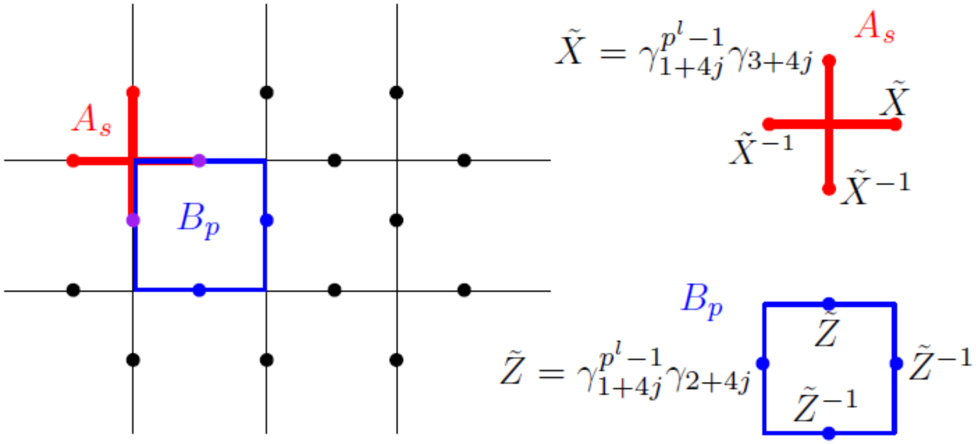
For qudit codes: **A. Ashikhmin and E. Knill, IEEE Trans. Inform. Theory 47, 3065 (2001)**

Let $D = p^{2l}$ where p is a prime number and $l \in \mathbb{Z}^+$. The operators

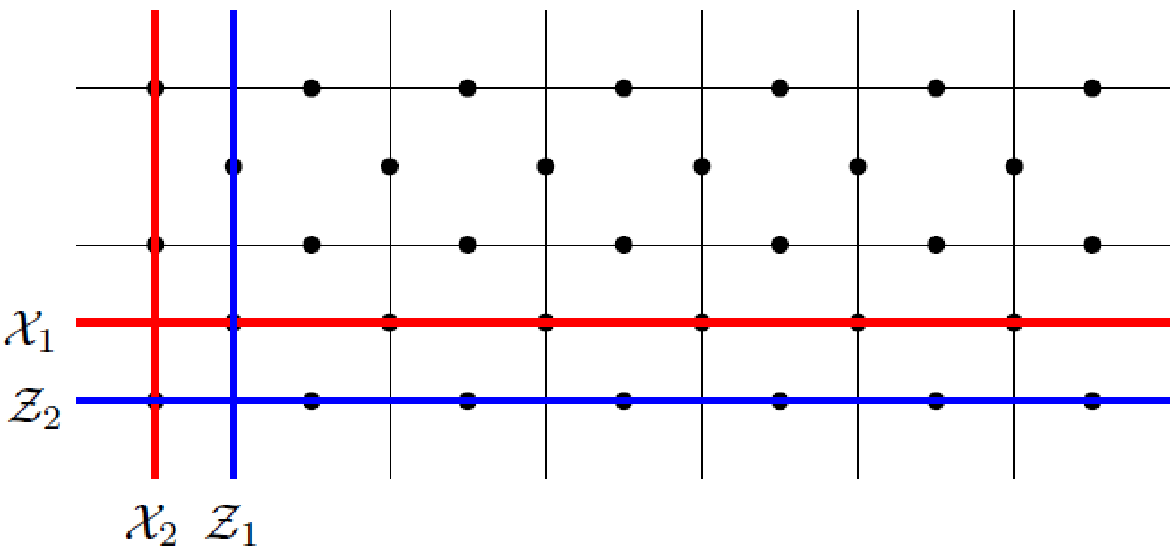
$$\tilde{Z}_{j+1} = \gamma_{1+4j}^{p^l-1}\gamma_{2+4j}, \qquad \tilde{X}_{j+1} = \gamma_{1+4j}^{p^l-1}\gamma_{3+4j},$$
$$\tilde{Q}_{j+1} = \gamma_{1+4j}^{\dagger}\gamma_{2+4j}^{\dagger}\gamma_{3+4j}\gamma_{4+4j}.$$

define a mapping of four parafermion modes onto a single qudit via the one-qudit stabilizer group $\mathcal{S}_{PF} = \langle \tilde{Q}_j \rangle$ and its corresponding logical operators $\mathcal{L}(\mathcal{S}_{PF}) = \langle \tilde{X}_j, \tilde{Z}_j \rangle$.

Mapping from qudits allowing odd logical operators



The parity of the horizontal (vertical) logical operators of the parafermion toric code is $a \times p^l$ $(b \times p^l)$ mod D.



Conclusions

- We define parafermion stabilizer codes which can be thought of as generalizations of Kitaev's chain model
- Local parafermion codes in general do not correspond to local qudit codes
- We construct parafermion toric code with adjustable protection against parity violating errors
- What can be said about finite temperature behavior?
  i.e. 1D Kitaev's chain – topological order at $T=0$,
  what about local models in 2D and 3D at $T>0$?