The advantages of qudit fault-tolerance

Earl T. Campbell (Sheffield)

Quantum Error Correction 2014 Zurich

Background image:
Computation Cloud
installation piece
Libby Heany

http://earltcampbell.com/research/

WHY QUDITS?

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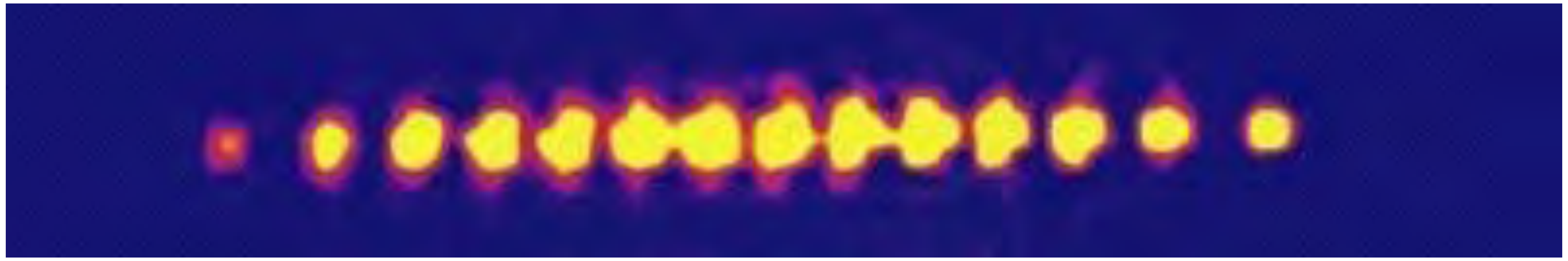
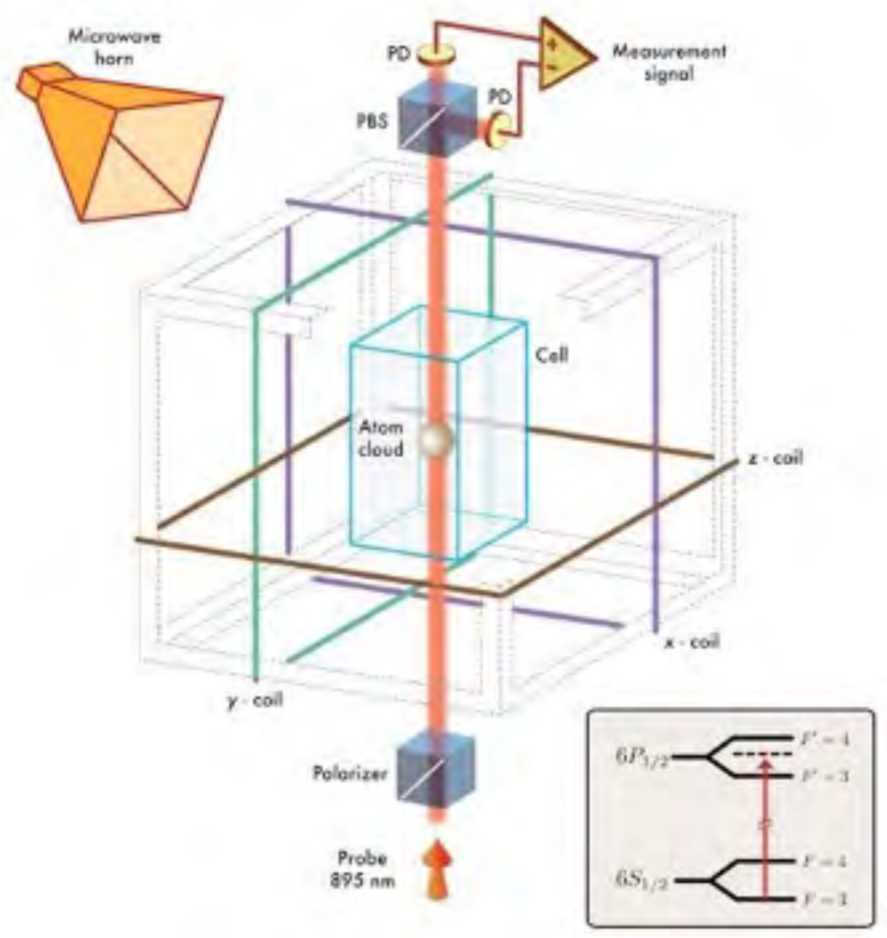
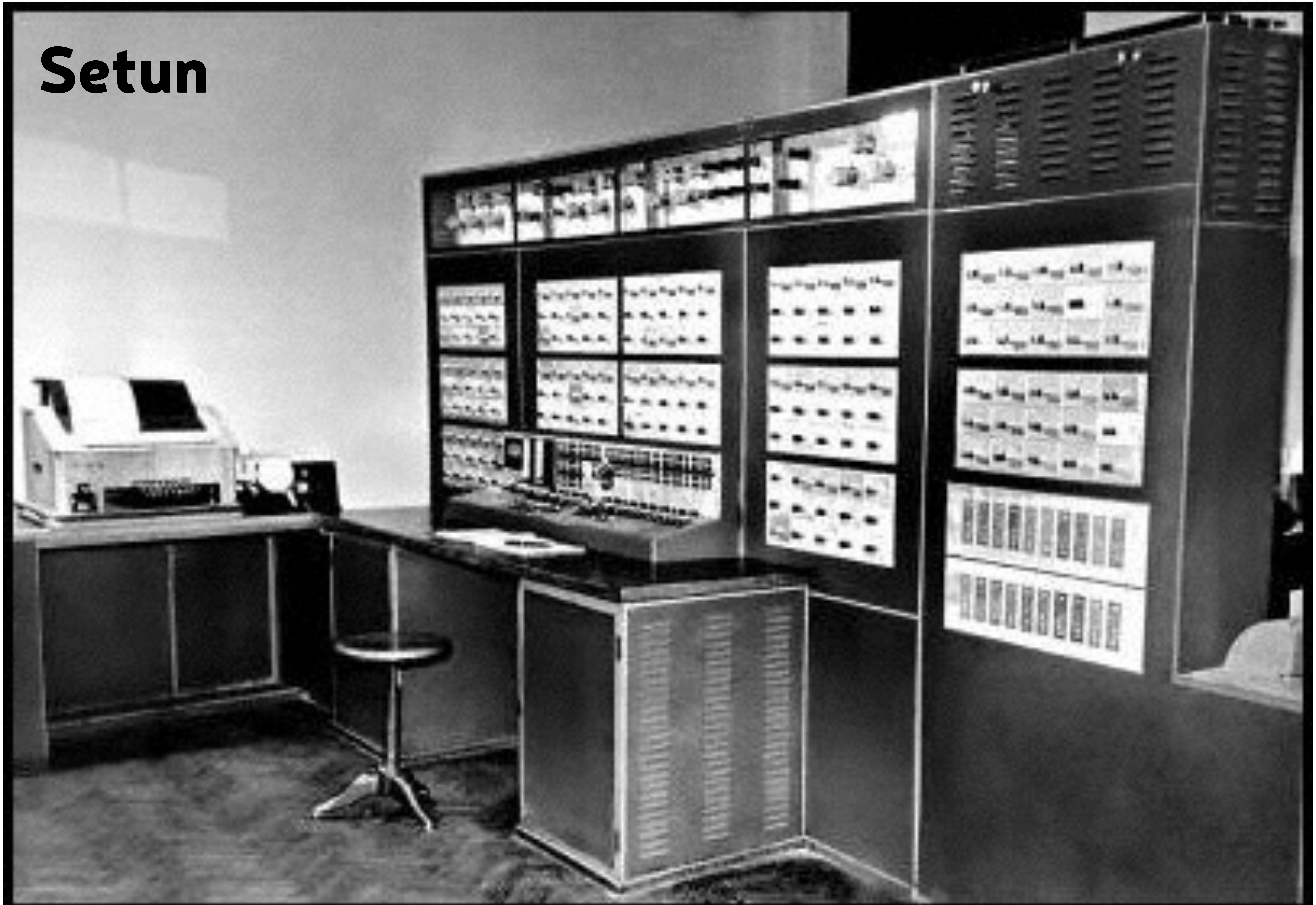


Monz et al. *Phys. Rev. Lett.* **106** 130506 (2011)



Smith et al.
Phys. Rev. Lett. **111**, 170502 (2013)
Anderson et al.
arXiv:1410.3891 (2014)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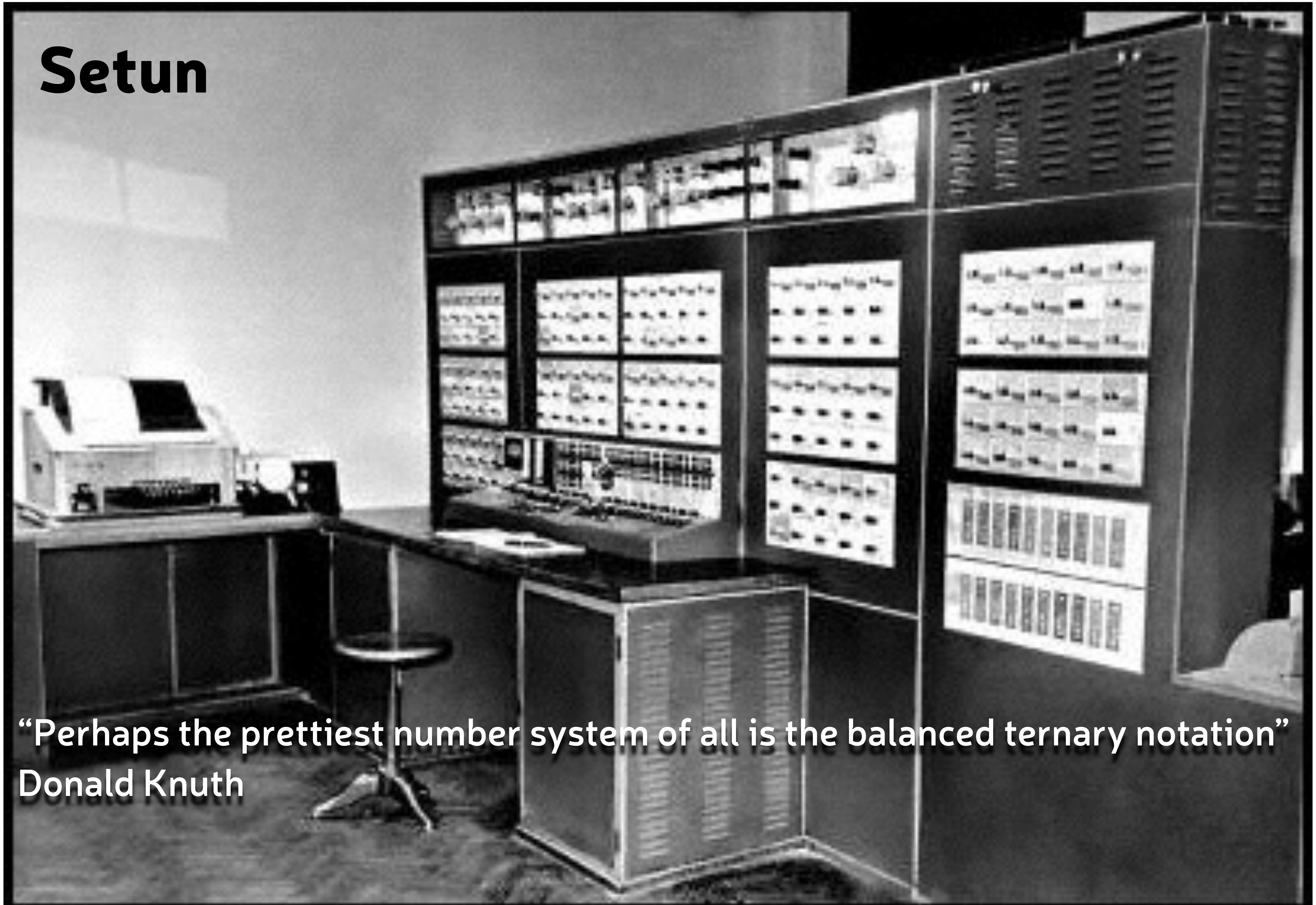

“Perhaps the prettiest number system of all is the balanced ternary notation”
Donald Knuth

OVERVIEW

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A contemporary approach to fault-tolerant quantum computing

STORAGE	MAGIC STATES	COMPILING	ALGORITHMS	PROFIT
Use QEC code to reduce noise.	**Prepare magic states and inject extra gates.**	**Compose finite gate set to produce any gate.**	**Implement quantum computation,**	**€ £ $**

ROUTES TO UNIVERSALITY

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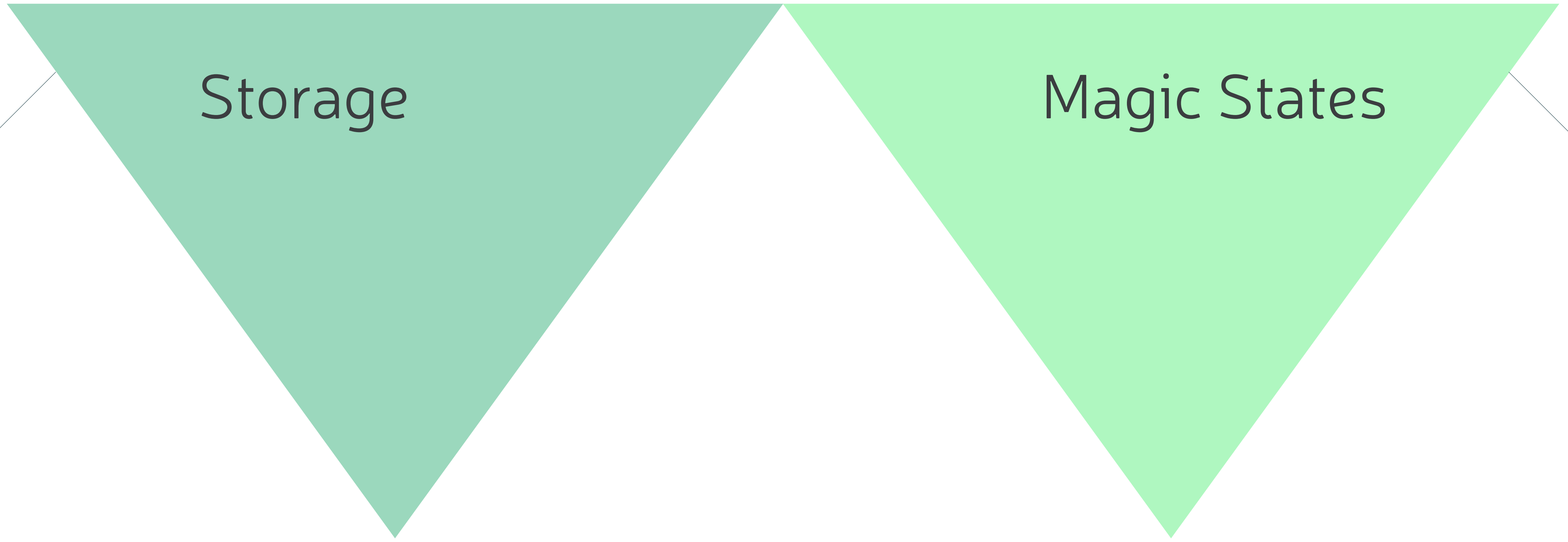
fault-tolerant ~ transversal

$$M_L \sim M \otimes M \otimes M \ldots$$



CODE 1

Efficient, high-threshold, allows fault-tolerant implementation of (most) Clifford group gates.

e.g. Toric code

CODE 2

Allows fault-tolerant implementation of a *non-Clifford gate,* e.g. pi/8 gate

e.g. Reed-Muller codes

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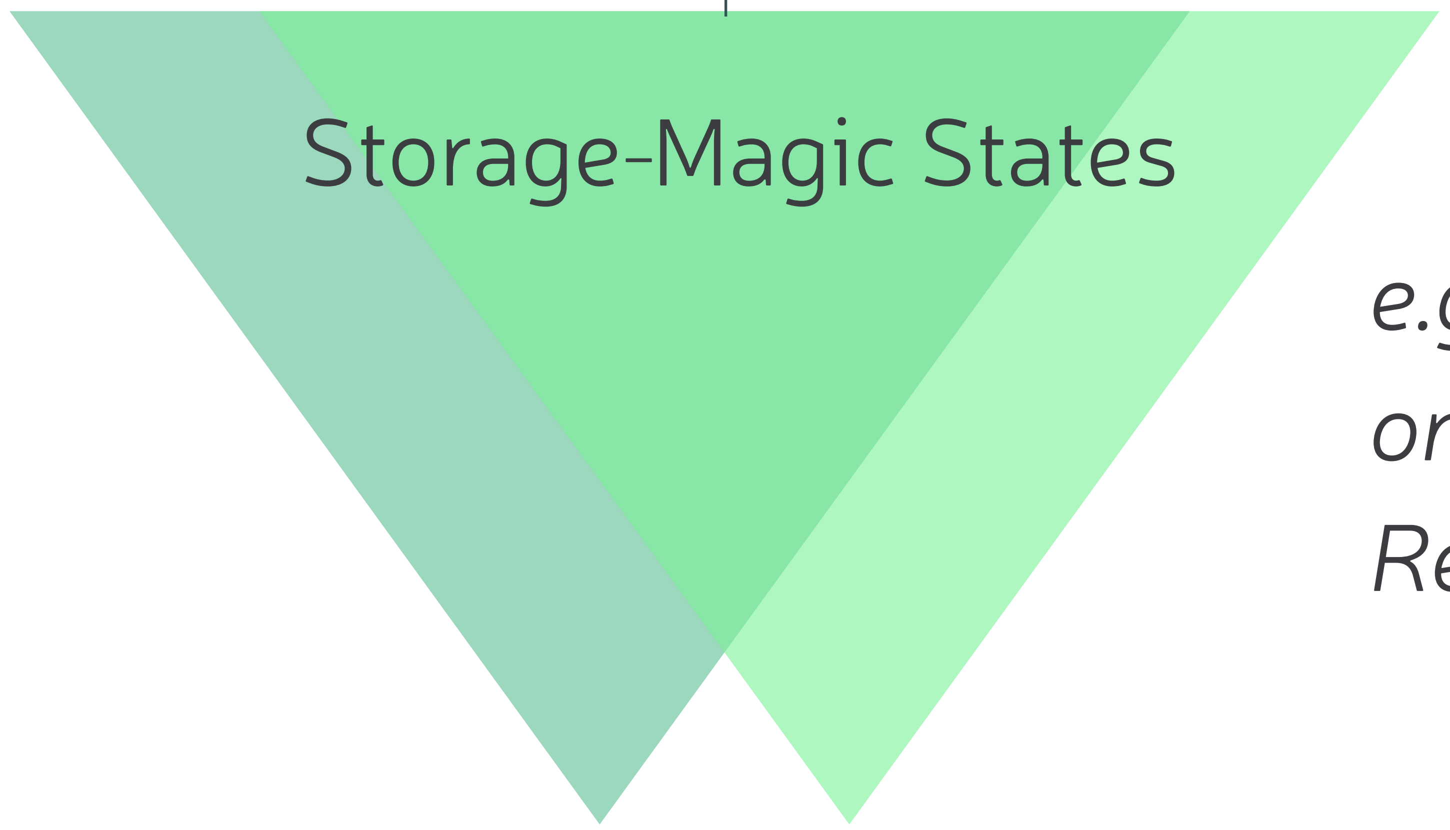
SUBSYSTEM CODES + GAUGE FIXING

Only 1 code. Potentially fewer resources needed.

But must *also* allow fault-tolerant non-Clifford gate.



e.g. gauge colour codes, or gauge variants of Reed-Muller codes

Many other alternative, but all rely on these exotic codes

A. Paetznick and B. W. Reichardt, *Phys. Rev. Lett.* **111**, 090505 (2013).

H. Bombin *et. al.*, *New J. Phys.* **15**, 055023 (2013)

J. T. Anderson *et. al.*, *Phys. Rev. Lett.* **113**, 080501 (2014).

T. Jochym-O'Connor and R. Laflamme, *Phys. Rev. Lett.* **112**, 010505 (2014).

OVERVIEW

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Qudit Toric Code

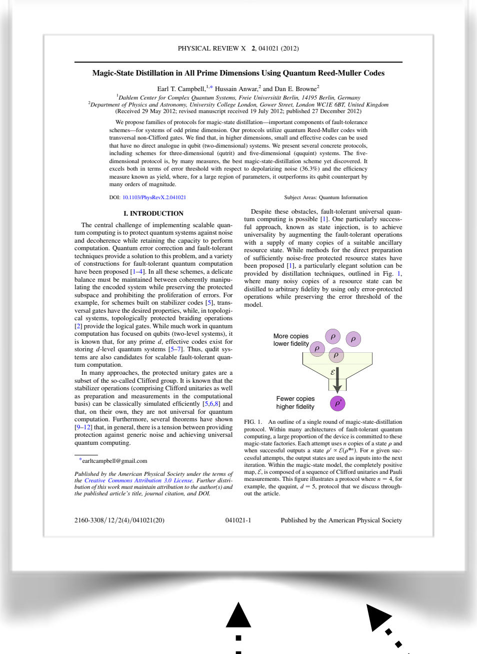
New J. Phys.

16 063038 (2014)

Qudit Magic

Phys. Rev. X

2 041021 (2012)

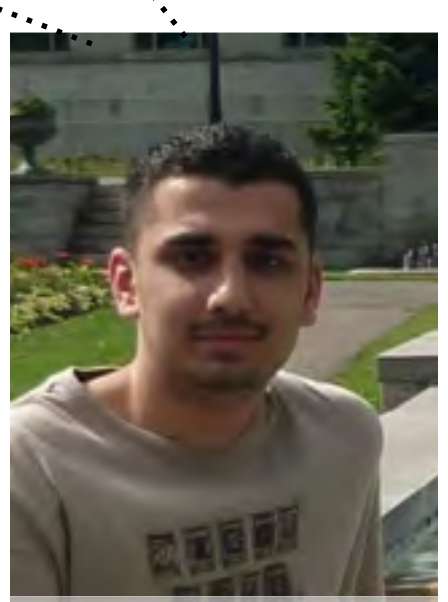


Phys. Rev. Lett

113 230501 (2014)

Benjamin
Brown

Dan
Browne



Hussain
Anwar

INTRODUCING QUDITS

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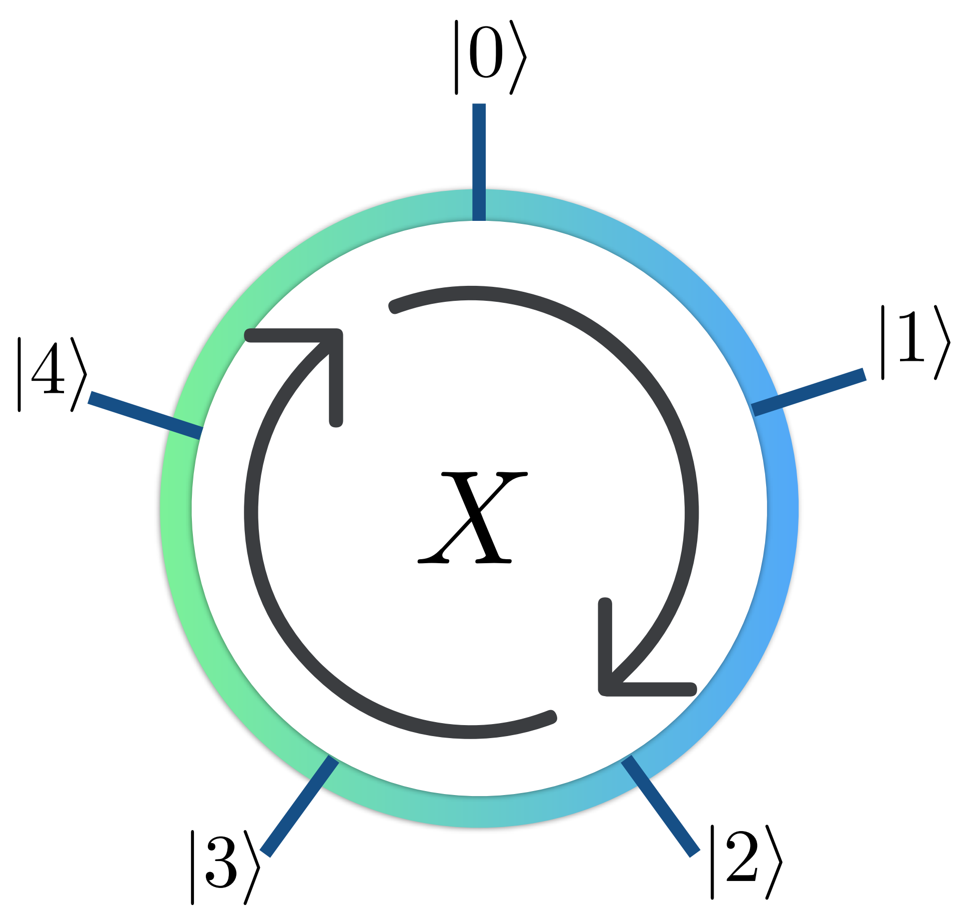
Define a basis $\{0, 1, \ldots, p-1\}$ with all arithmetic modulo *p*.

Imagine states as notches on a clock face.



Pauli-group $\mathcal{P}$**:**

generators
$$\begin{aligned} X|n\rangle &= |n+1\rangle \\ Z|n\rangle &= \omega^n |n\rangle \end{aligned}$$

where $\omega = e^{i2\pi/p}$

Clifford group $\mathcal{C}$ is normaliser of $\mathcal{P}$ so $C\mathcal{P}C^\dagger = \mathcal{P}$

Overcomplete set of generators

$$\begin{aligned} \mathcal{X}_{\alpha,\beta}|n\rangle &= |\alpha n + \beta\rangle \\ \mathcal{Z}_{\alpha,\beta}|n\rangle &= \omega^{\alpha n + \beta n^2}|n\rangle \end{aligned} \qquad \begin{aligned} H|n\rangle &= \tfrac{1}{\sqrt{p}} \textstyle\sum_m \omega^{nm}|m\rangle \\ C_X|n\rangle|m\rangle &= |n\rangle|m+n\rangle \end{aligned}$$

Assuming **p** is an odd prime!

QUDIT TORIC CODES

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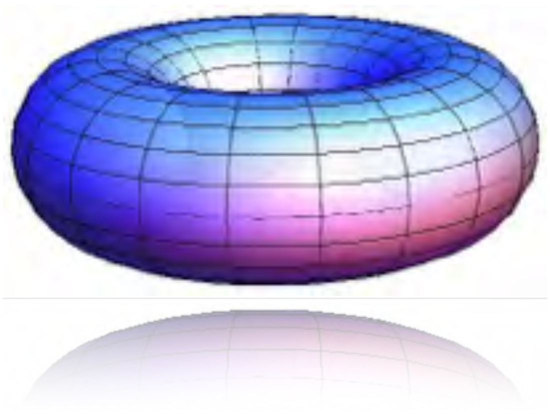
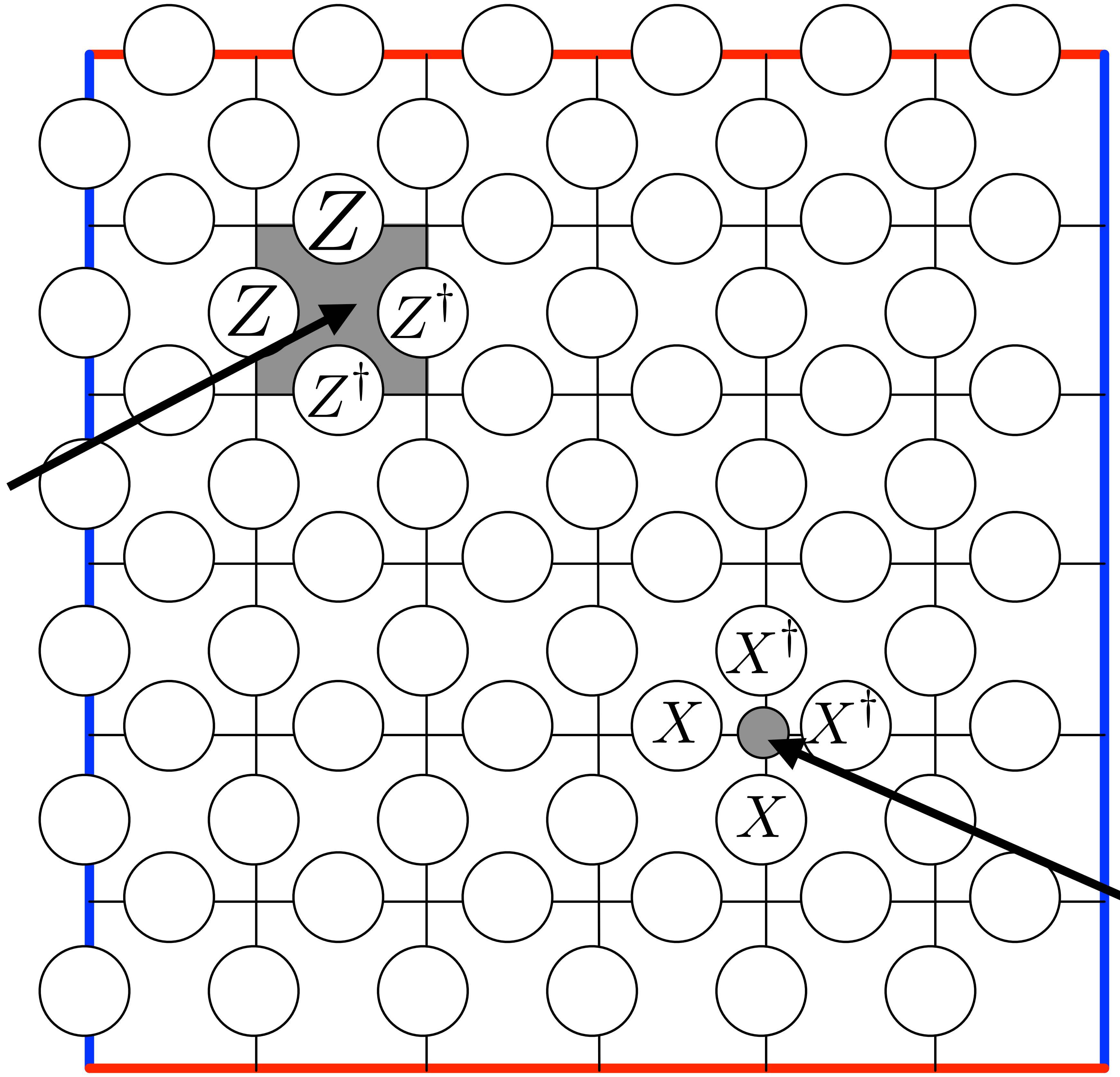


The toric code straightforwardly generalizes

though we have to adjust the code slightly



plaquettes

stars

Stabilizers

are still 4-body terms
but “daggerized”

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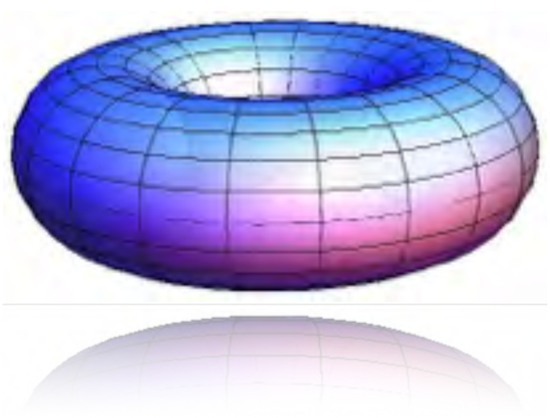

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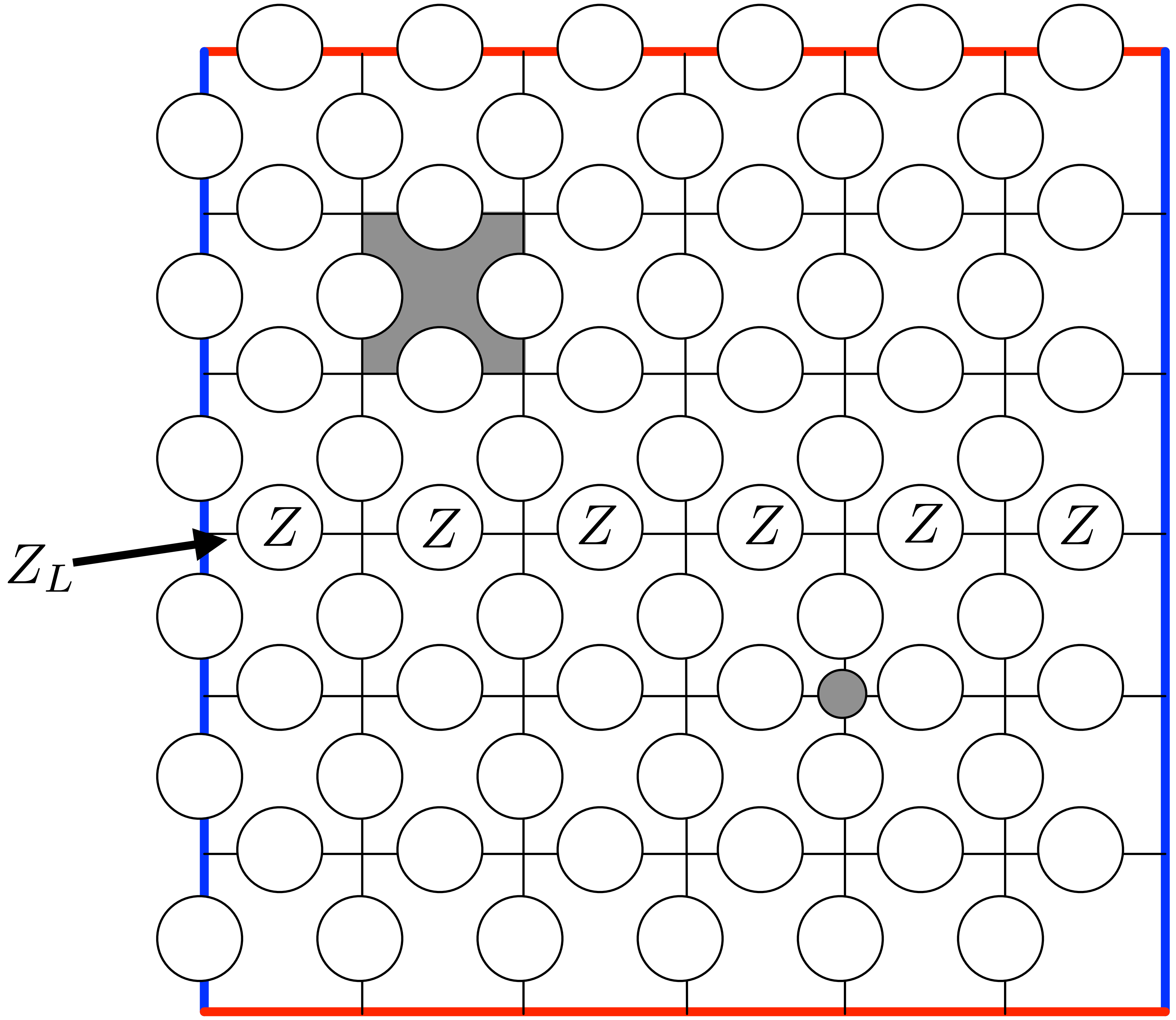
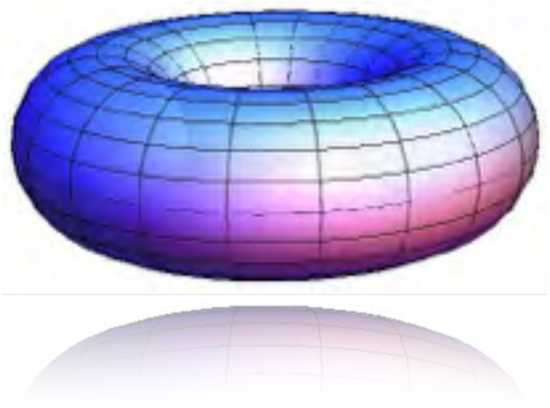


Logical operators

are still closed loops
(in the homological sense)



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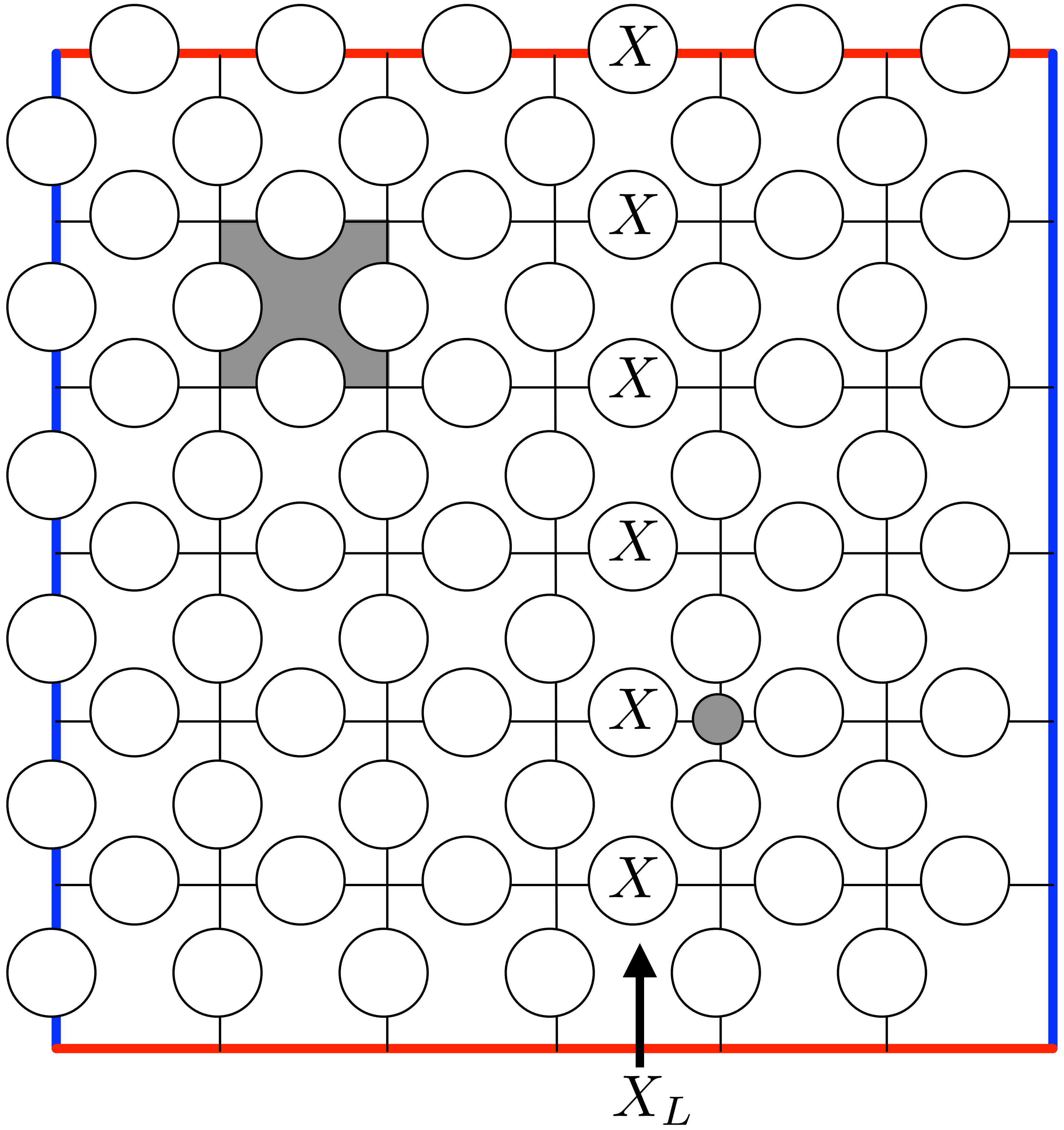

Logical operators
are still closed loops
(in the homological sense)

the shortest loop is length
in a code of $2L^2$ qudits.

They are $[[2L^2, 2, L]]$ codes.

DECODERS

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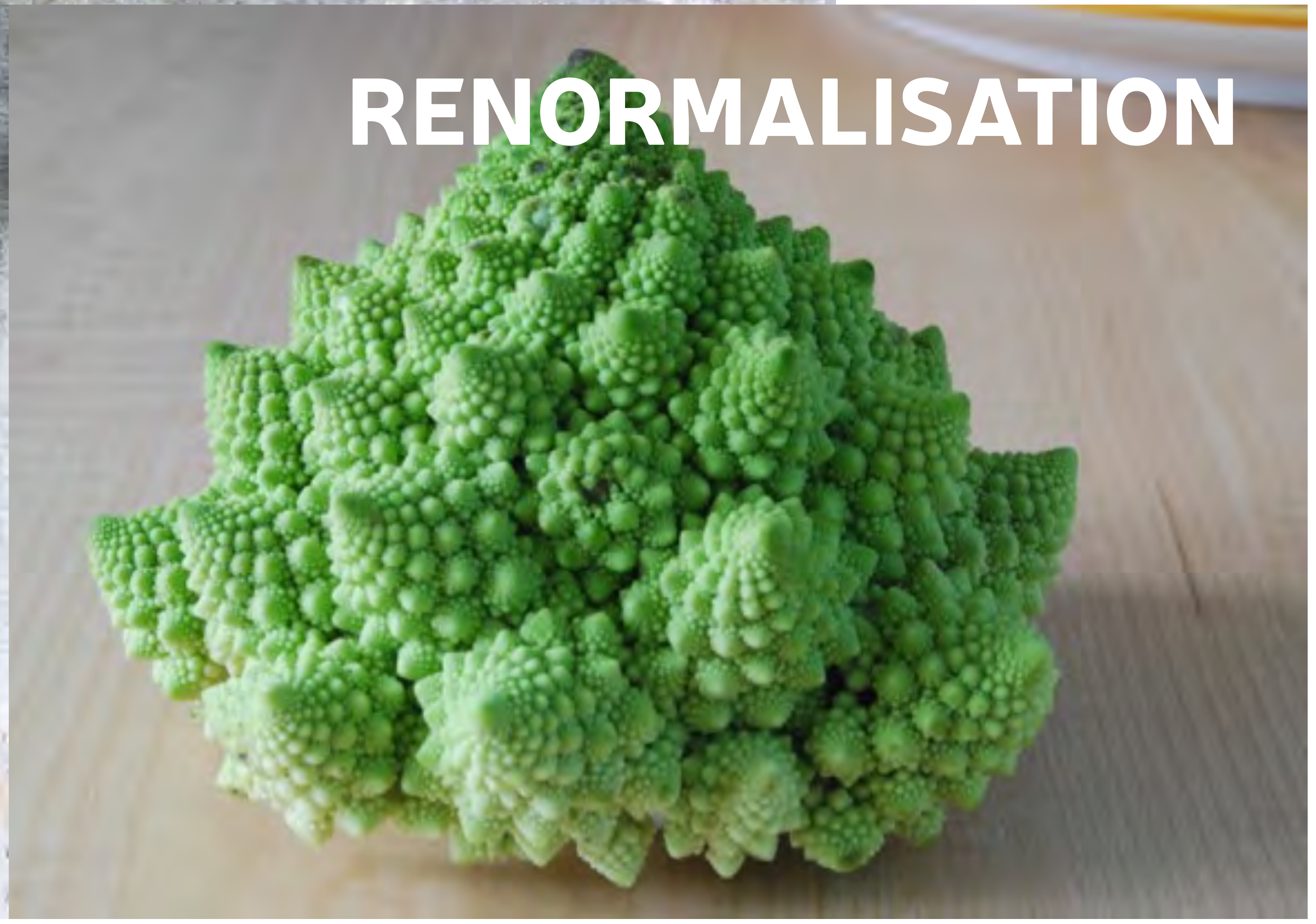


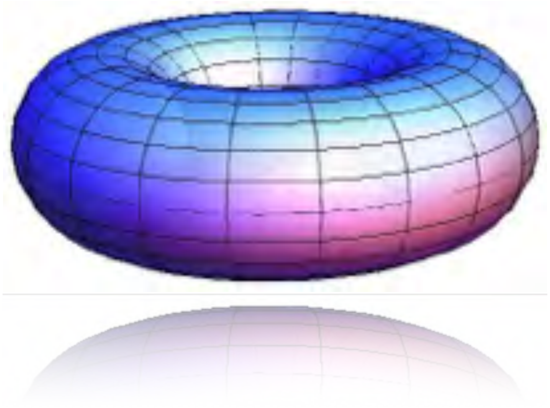
S.Bravyi, J, Haah

Phys. Rev. Lett. **111,** 200501

G. Duclos-Cianci, D. Poulin.,

Phys. Rev. Lett. **104**, 050504

RESULTS

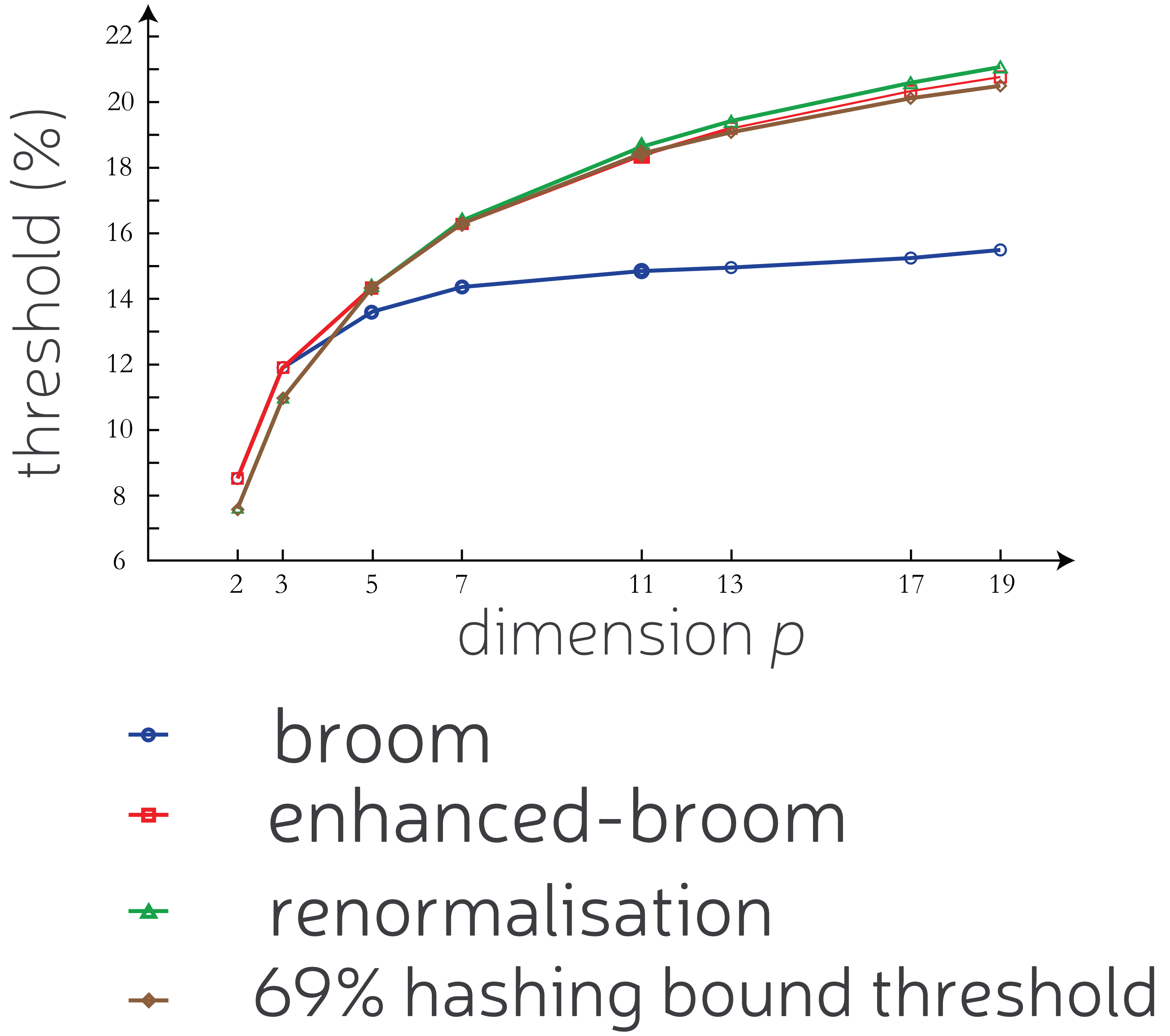


Decoding and Thresholds

- Thresholds improve with p;
- Thresholds follow ~69% of the hashing bound threshold;
- We need to tweak broom to get this.

- Similar results in:
  - G. Duclos-Cianci, D. Poulin., *Phys. Rev. A* **87** 062338
  - R. Andrist, J. Wootton, H. Katzgraber., arXiv:1406.5974

- Advantage persists in when accessing for noisy stabiliser measurement. Thresholds up to 8%
  - F. Watson, H. Anwar, D. Browne *arXiv:1411.3028*



H. Anwar, *et al New J. Phys.* **16** 063038 (2014)

Magic States Pt. 1

Implementing non-Clifford gates

Good examples of non-Clifford gates

Qubit

$$U|n\rangle = \nu^n |n\rangle$$

with $\nu = \exp\left[i\frac{2\pi}{2^3}\right]$

Qutrit (p=3)

$$V|n\rangle = \tau^n |n\rangle$$

with $\tau = \exp\left[i\frac{2\pi}{3^2}\right]$

Qudits (p>3)

$$M|n\rangle = \omega^{n^3} |n\rangle$$

with $\omega = \exp\left[i\frac{2\pi}{p}\right]$

M. Howard, J. Vala., *Phys. Rev. A* **86**, 022316 (2012)
E. Campbell, H. Anwar, D. Browne., *Phys. Rev. X* **2** 041021 (2012)

Good examples of non-Clifford gates

Qubit

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Qutrit (p=3)

$$V|n\rangle = \tau^n |n\rangle$$

with $\tau = \exp\left[i\frac{2\pi}{3^2}\right]$

Qudits (p>3) are CUBIC gates

$$M|n\rangle = \omega^{n^3} |n\rangle$$

with $\omega = \exp\left[i\frac{2\pi}{p}\right]$

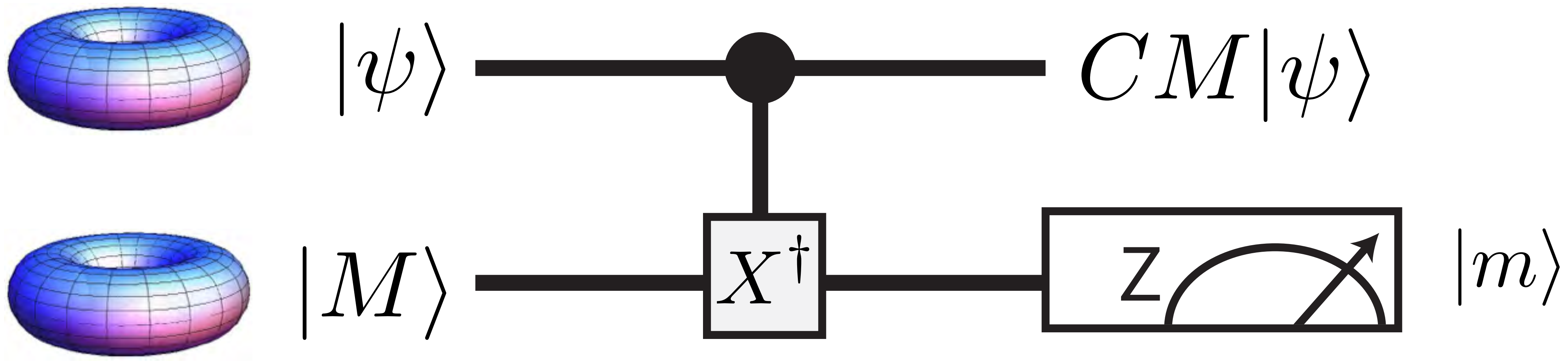
Clifford hierarchy 3rd level

Conjugates Pauli group to Clifford group

$$M\mathcal{P}M^\dagger = \mathcal{C}$$

enables gate-injection using resource of magic-states

$$|M\rangle = M|+\rangle = \frac{1}{\sqrt{p}}\sum_n \omega^{n^3}|n\rangle$$



where

$$C = MX^m M^\dagger$$

MAGIC STATE DISTILLATION

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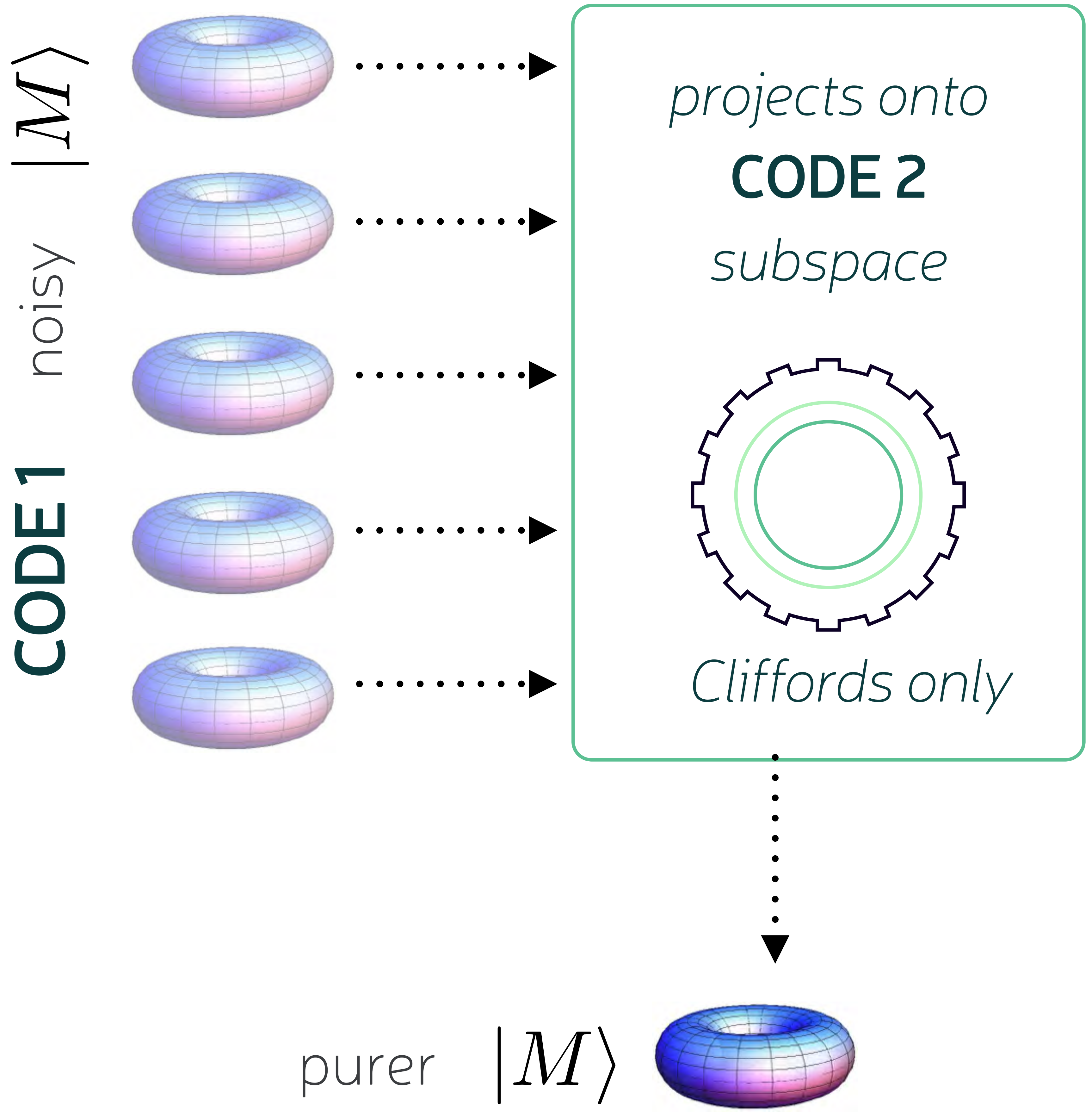

Given an [[n,k,d]] **code with transversal M**

for noise rate $\epsilon = 1 - \langle M|\rho|M\rangle$ we find

- A threshold: if $\epsilon < \epsilon^*$ then $\epsilon_{\text{out}} < \epsilon$
- Exponential: $\epsilon_{\text{out}} < C\epsilon^d$
- Overhead: $N \leq A_\epsilon \log^\gamma(\epsilon_{\text{target}})$

where we use ***N*** raw copies, and

$$\gamma = \log(n/k)/\log(d)$$

S. Brayvi and A. Kitaev., *Phys. Rev. A* **71**, 022316 (2005)
E. Campbell, H. Anwar, D. Brown., *Phys. Rev. X* **2** 041021 (2012)

TRIORTHGONALITY

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Define a matrix $G = \begin{pmatrix} G_0 \\ \hline G_1 \end{pmatrix}$ r, k, n

Logical states $|\vec{0}_L\rangle = \frac{1}{\sqrt{p^r}} \sum_{u \in \mathrm{span}(G_0)} |u\rangle$

$$|\vec{m}_L\rangle = \frac{1}{\sqrt{p^r}} \sum_{u \in \mathrm{span}(G_0)} |u + G_1 \vec{m}\rangle$$

Define

$$X[\vec{v}] = X^{v_1} \otimes X^{v_2} \otimes \ldots X^{v_n}$$

stabilizer generators $X[\vec{v}]$ for all $\vec{v} \in G_0$

logical operators $X[\vec{v}]$ for all $\vec{v} \in G_1$

Definition

A matrix/code is said to be *triorthgonal* if for all $v, v', v'' \in G$

whenever $v = v' = v''$ $\sum_j v_j^3 = \begin{cases} 0 \pmod{p} & , v \in G_0 \\ a \neq 0 \pmod{p} & , v \in G_1 \end{cases}$ otherwise, $\sum_j v_j v'_j v''_j = 0 \pmod{p}$

Theorem

A triorthgonal code has a transversal non-Clifford U, V or M (upto a Clifford)

qubits S. Brayvi and J. Haah., Phys. Rev. A **86**, 052329 (2012)

qudits *unpublished / in preparation*

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Triorthogonal codes

Finding the exotic codes

Reed-Muller codes are $[[p^m - 1, 1, ?]]$ codes where m is the order.
Simplest qubit triorthgonal code in the "Reed-Muller" family is order 4.

$$\begin{array}{c} v \\ v' \\ v'' \\ \\ \\ \end{array} \left(\begin{array}{ccccccccccccccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ \hline 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{array}\right)$$

$$U|n\rangle = \nu^n |n\rangle$$

$$\nu = \exp\left[i\frac{2\pi}{2^3}\right]$$

$$v \cdot v' \cdot v'' = (\ 1\ \ 1\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ )$$

$$\sum_j v_j \cdot v'_j \cdot v''_j = 2 = 0 \pmod 2$$

Definition

A matrix/code is said to be triorthgonal if for all $v, v', v'' \in G$

whenever $v = v' = v''$ $\quad \sum_j v_j^3 = \begin{cases} 0 \pmod p & , v \in G_0 \\ a \neq 0 \pmod p & , v \in G_1 \end{cases}$ otherwise, $\sum_j v_j v'_j v''_j = 0 \pmod p$

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Triorthogonal codes

Finding the exotic codes

Simplest qudit “Reed-Muller” code for dimension 3

$$\left(\begin{array}{cccccccc} 0 & 0 & 1 & 1 & 1 & 2 & 2 & 2 \\ 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 \\ \hline 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{array}\right)$$

$$V|n\rangle = \tau^n |n\rangle$$

with $\tau = \exp\left[i\frac{2\pi}{3^2}\right]$

an $[[8,1,2]]$ code so order 2

Compared to 15 qubit:

1. smaller code;
2. worse distance;
3. worse overhead (gamma);
4. threshold?

QUTRITS: THRESHOLDS

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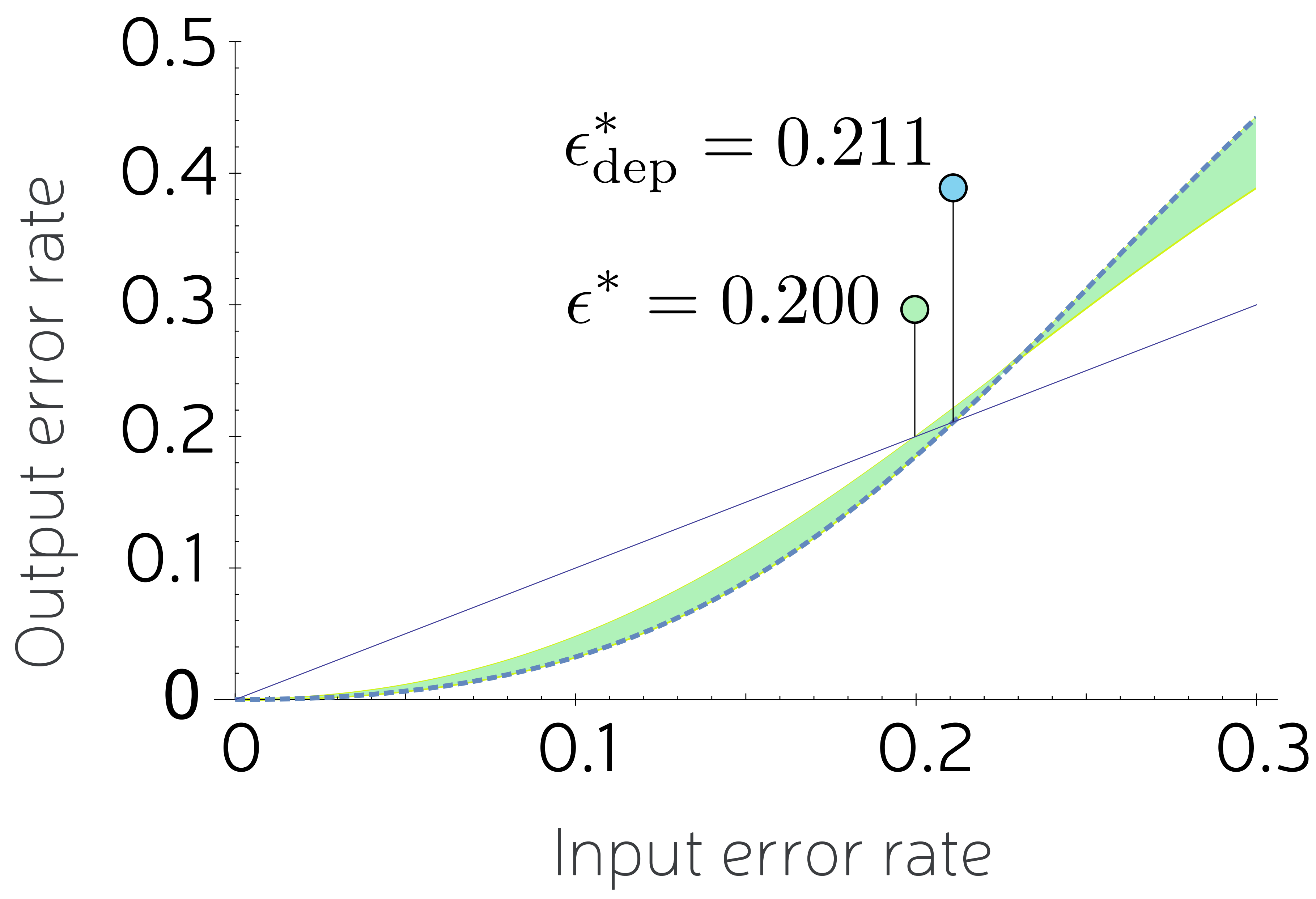
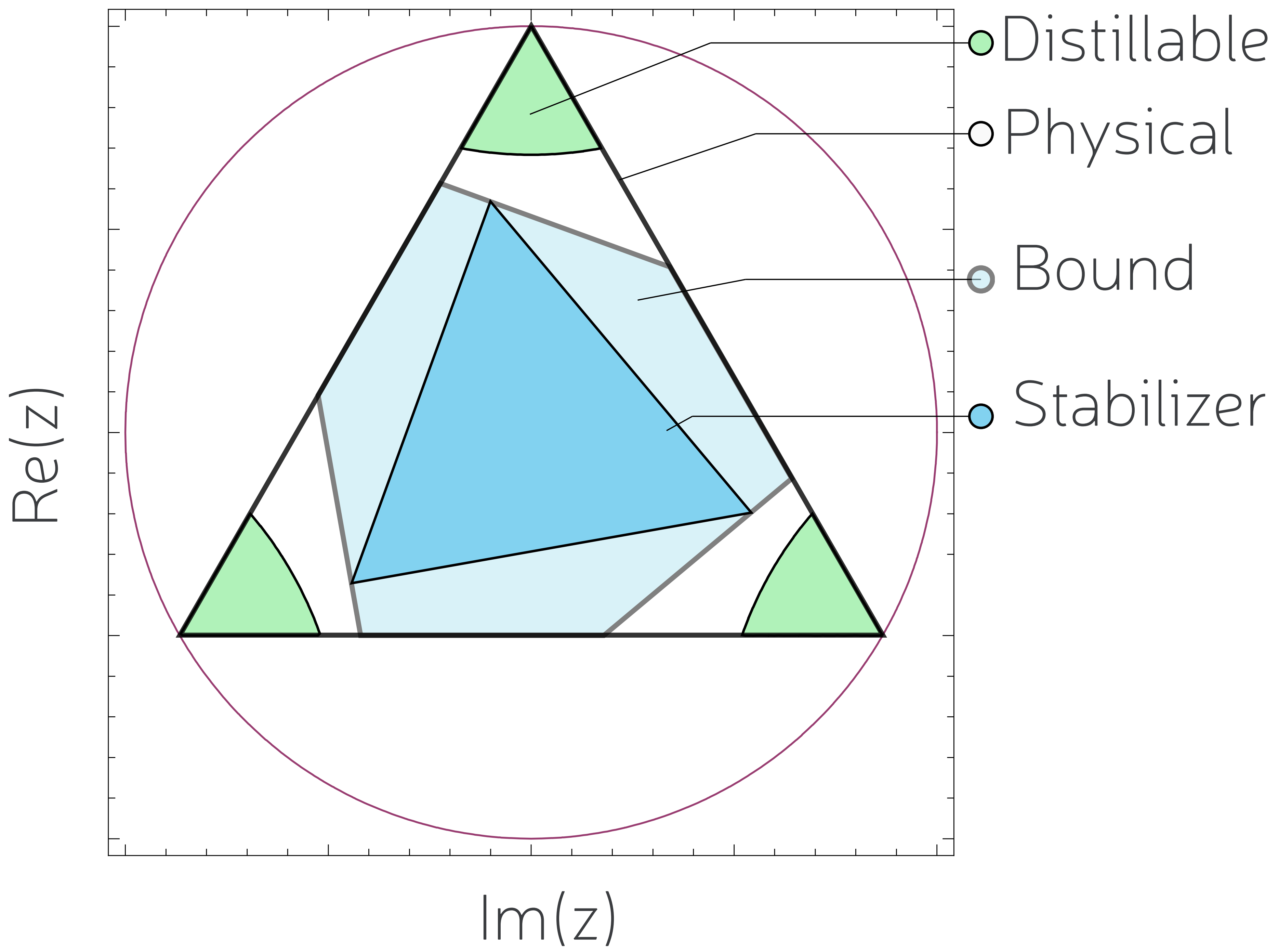
In higher dimensions,
$\epsilon = 1 - \langle M|\rho|M\rangle$
doesn't uniquely identify the state.

Let us look a slice of state space (which we can always project onto)
We parameterize by $z = \mathrm{tr}[M\rho]$





Veitch et. al. *New Journal of Physics* **14** 113011 (2012)

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Triorthogonal codes

Finding the exotic codes

Simplest qudit “Reed-Muller” code

for dimension 5

$$\left(\begin{array}{cccc} 1 & 2 & 3 & 4 \\ \hline 1 & 1 & 1 & 1 \end{array}\right)$$

$$M|n\rangle = \omega^{n^3}|n\rangle$$

with $\omega = \exp\left[i\frac{2\pi}{p}\right]$

for dimension 7

$$\left(\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & 1 & 1 & 1 & 1 & 1 \end{array}\right)$$

for dimension 11

$$\left(\begin{array}{cccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ \hline 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{array}\right)$$

Gives a triorthonormal family of order 1 Reed-Muller codes.

$[[p-1, 1, 2]]$

PERFORMANCE

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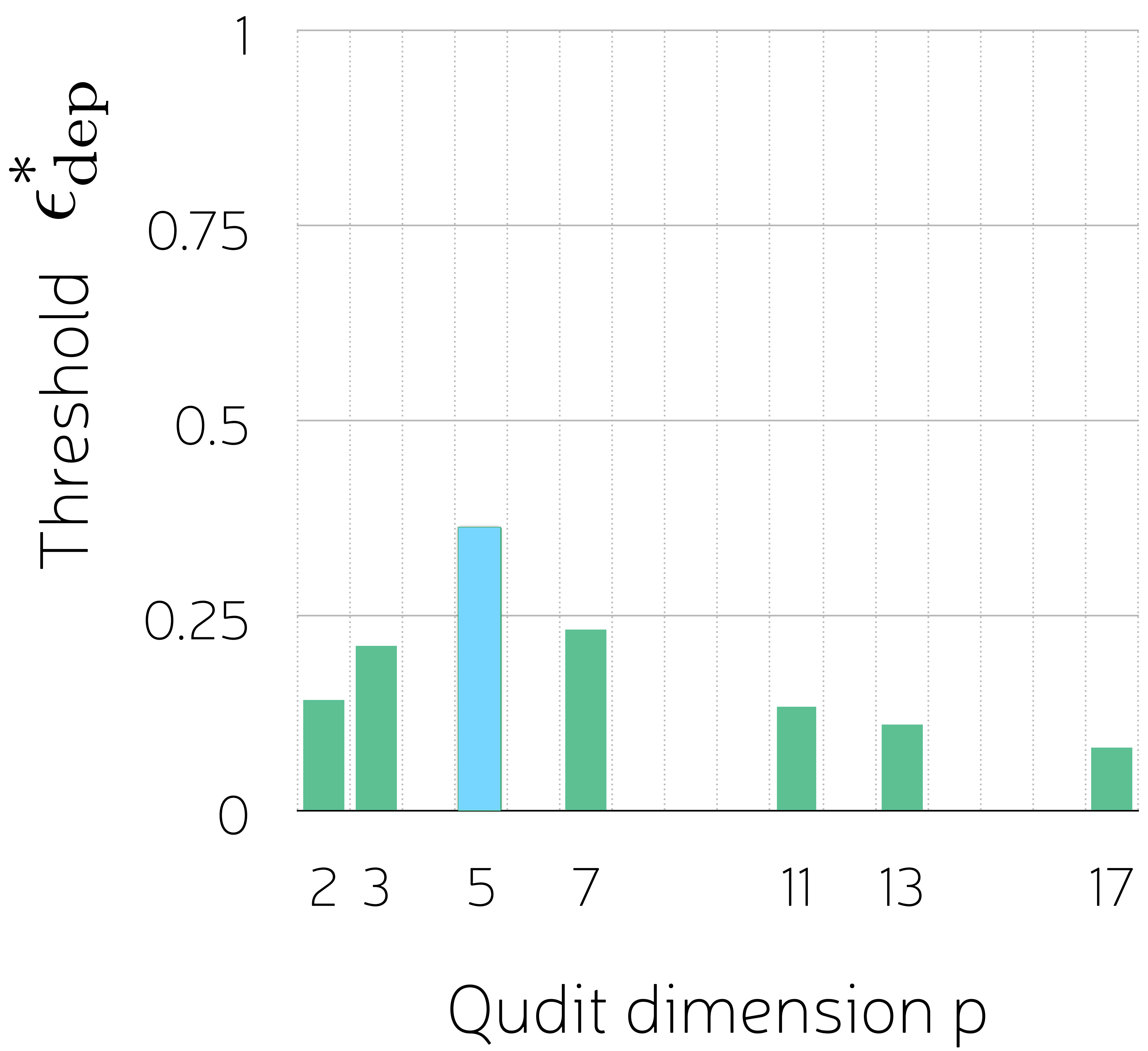
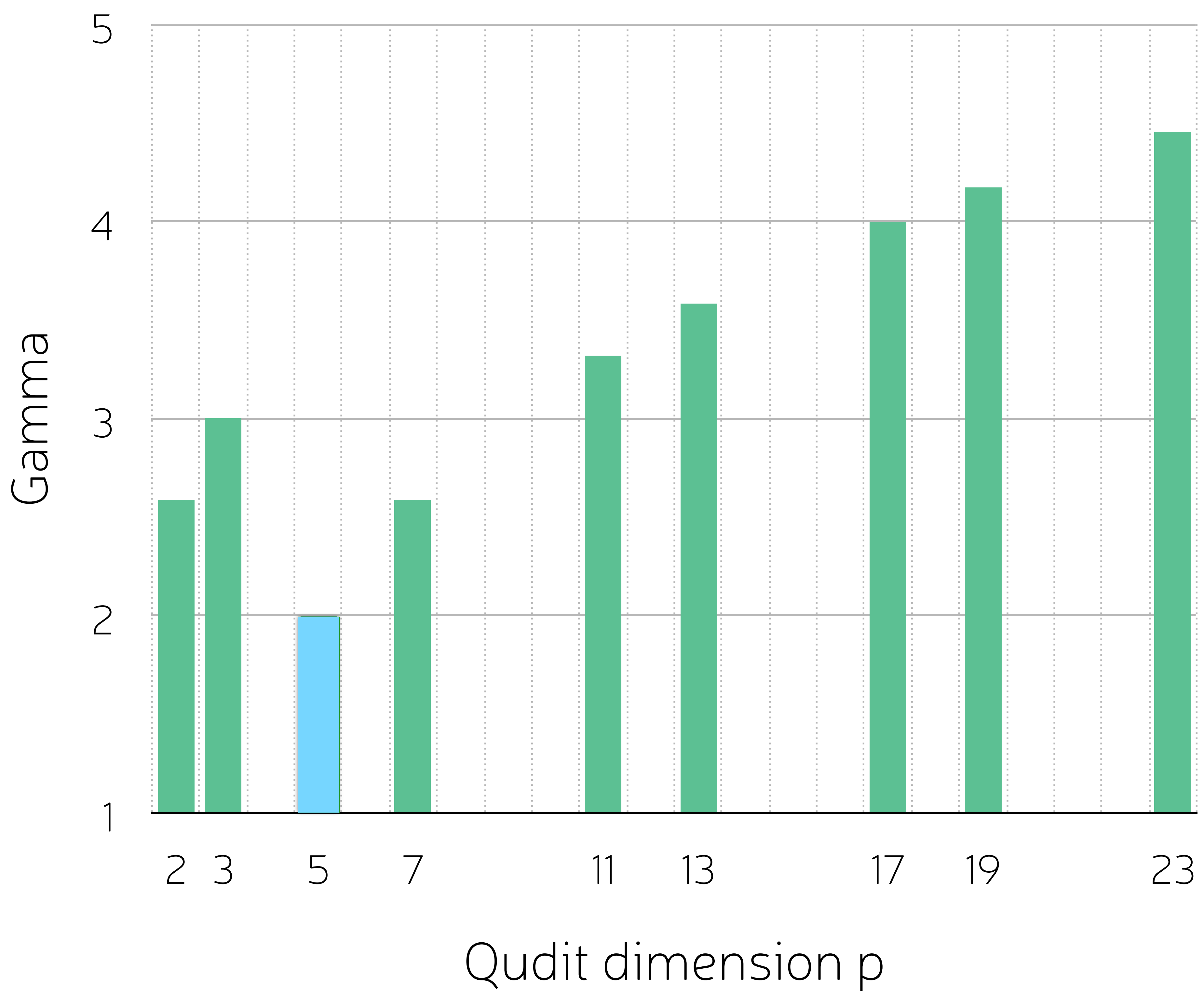
Threshold



Overhead



remember

$$N \leq A_\epsilon \log^\gamma(\epsilon_{\mathrm{target}})$$

Magic States pt 2

Beyond linear functions

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Reed-Muller codes in more detail

- stick to first-order codes $[[(p^m-1),1,?]]$
- define generating matrix in terms of functions

$$G=\left(\frac{G_0}{G_1}\right)=\left(\begin{array}{cccc} f_1(1) & f_1(2) & \dots & f_1(p-1) \\ f_2(1) & f_2(2) & \dots & f_2(p-1) \\ & & \vdots & \\ f_r(1) & f_r(2) & \dots & f_r(p-1) \\ \hline 1 & 1 & \dots & 1 \end{array}\right)$$

- earlier we had only 1 function $f_1(x)=x$
- a *degree-r Reed-Muller code* to has generating functions

$$f_1(x)=x, f_2(x)=x^2, \dots f_r(x)=x^r$$

Reed-Muller codes in more detail

- a *degree-r Reed-Muller code* to has generating functions

$$f_1(x) = x,\ f_2(x) = x^2,\ f_r(x) = x^r$$

Result

A degree-r Reed-Muller code is triorthgonal if

$$r < (p-1)/3$$

Furthermore, it can detect up-to r errors.

1. consider a triple-product from G_0

$$g(x) = f_a(x) f_b(x) f_c(x) = x^q$$
$$q = a + b + c$$

2. for prime numbers we know

$$\sum_x x^q \pmod{p} = \begin{cases} 0 & q \neq 0 \pmod{p-1} \\ p-1 & q = 0 \pmod{p-1} \end{cases}$$

3. triorthgonality ensured if $\forall a, b, c \leq r$ we have $a + b + c < p - 1$

E. Campbell. *Phys. Rev. Lett* **113** 230501 (2014)

PERFORMANCE

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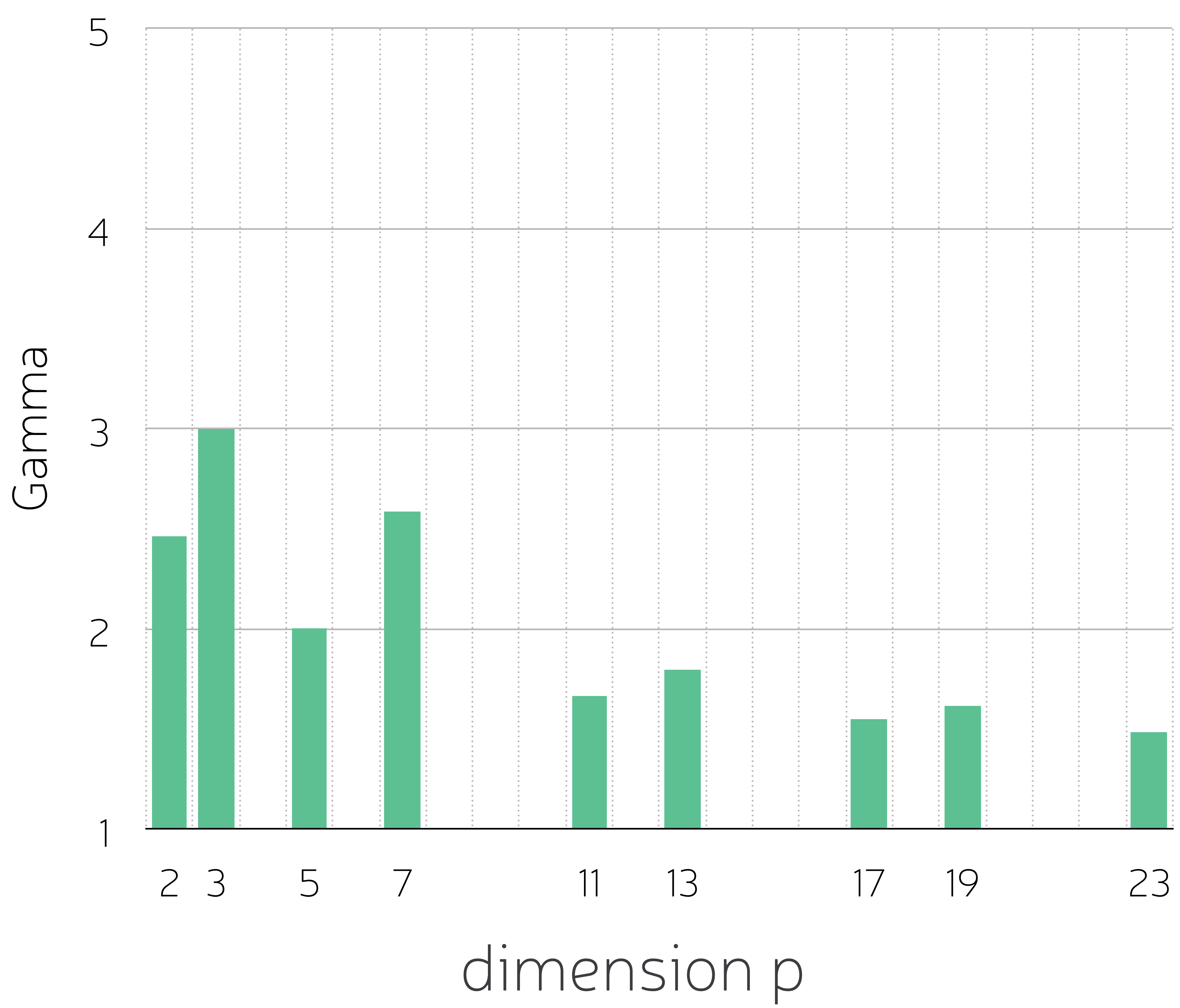
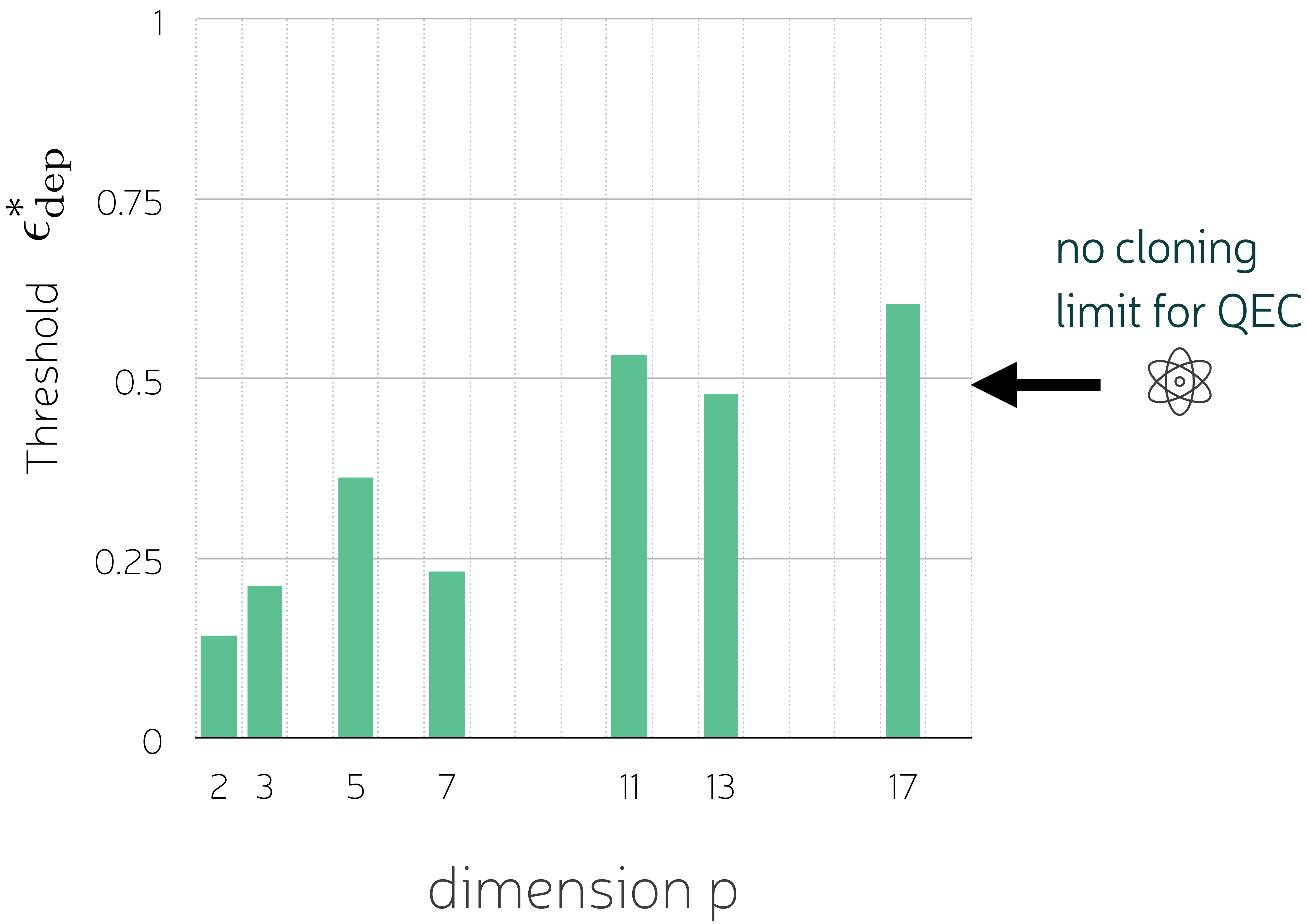
Overhead



Threshold



PERFORMANCE

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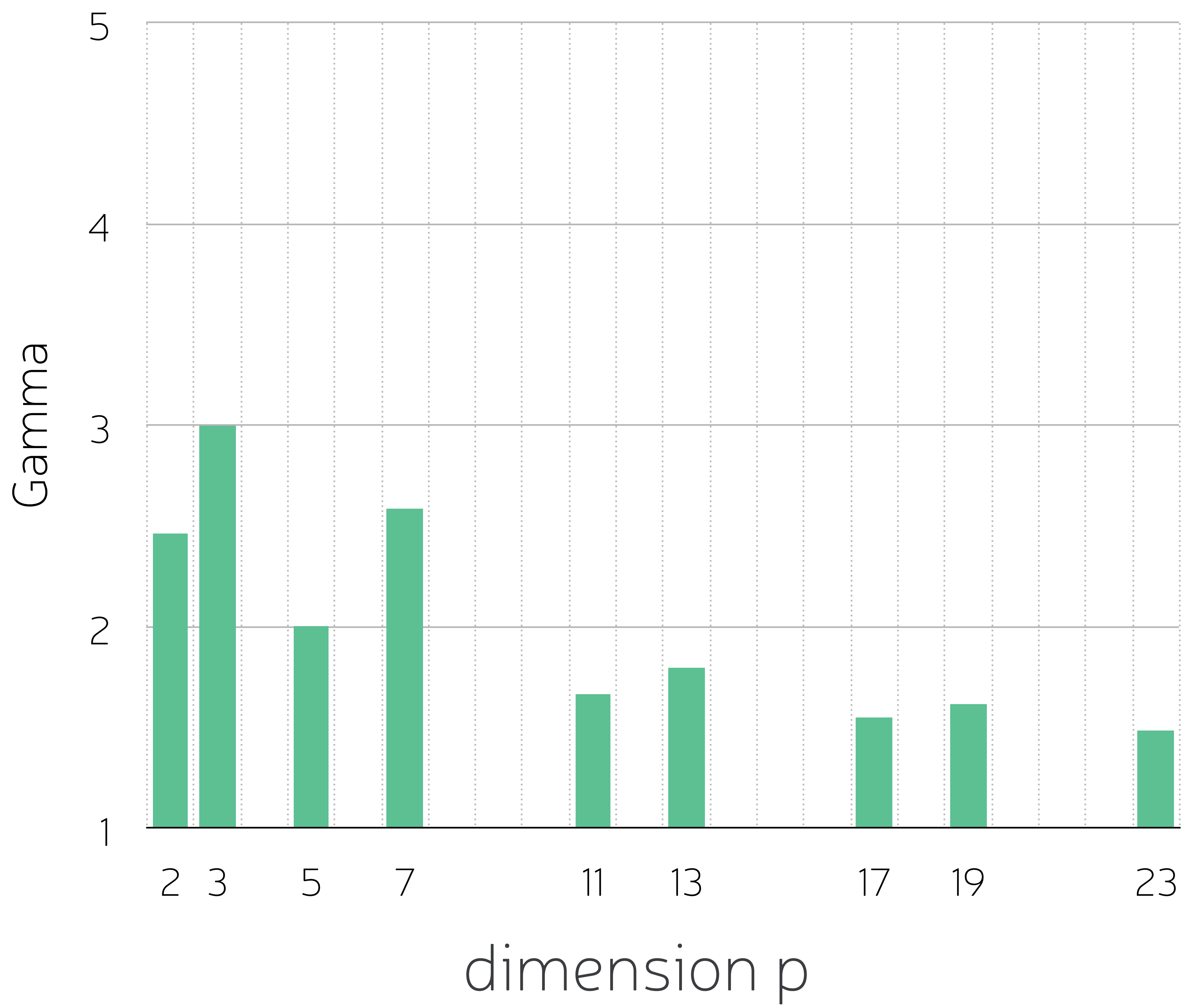
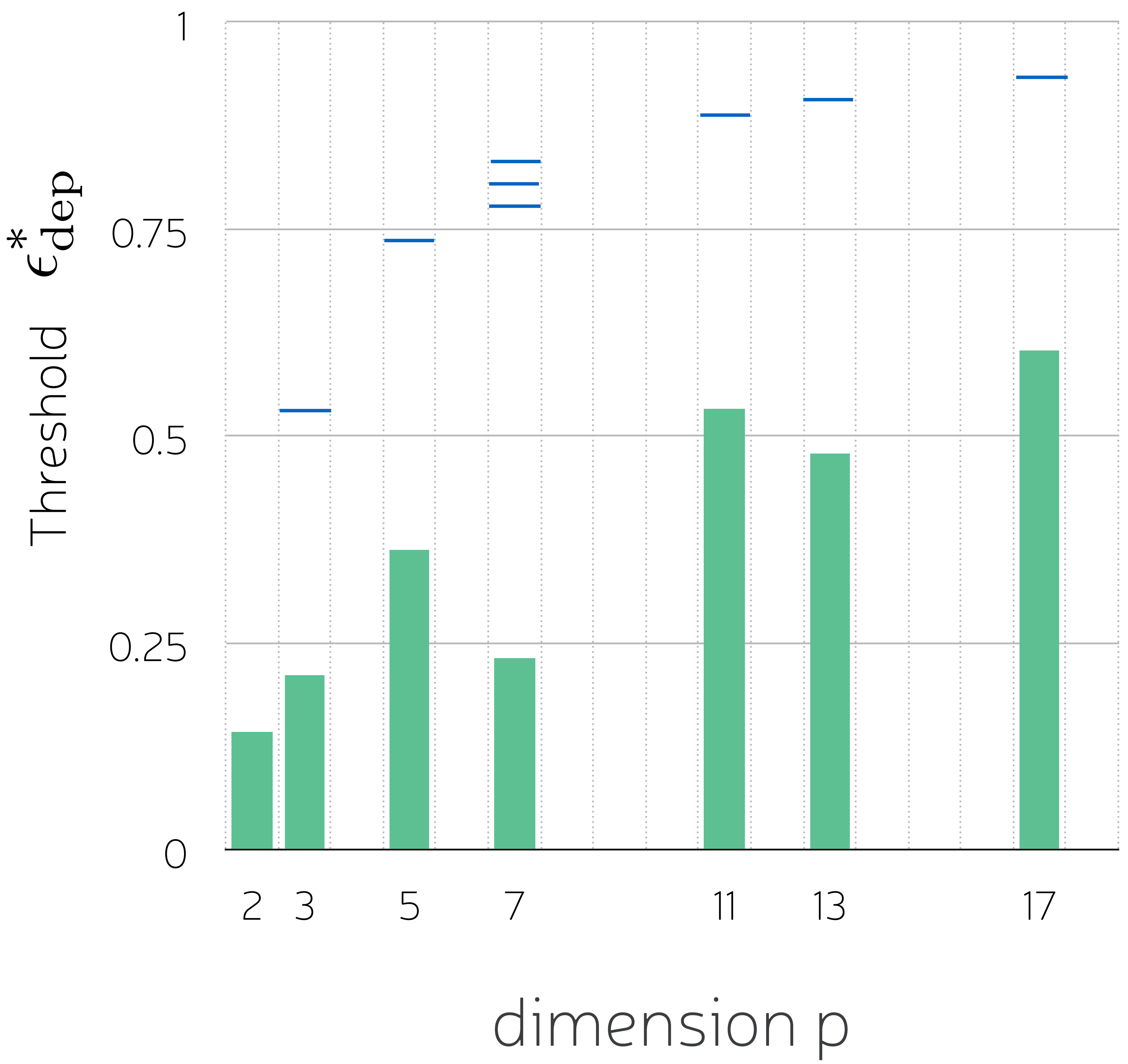
Overhead



Threshold



I. Bengtsson, K. Blanchfield, E. Campbell, M. Howard
J. Phys. A: Math. Theor. **47** 455302 (2014)
appendix of E. Campbell. *Phys. Rev. Lett* **113** 230501 (2014)

GRAND COMPARISON

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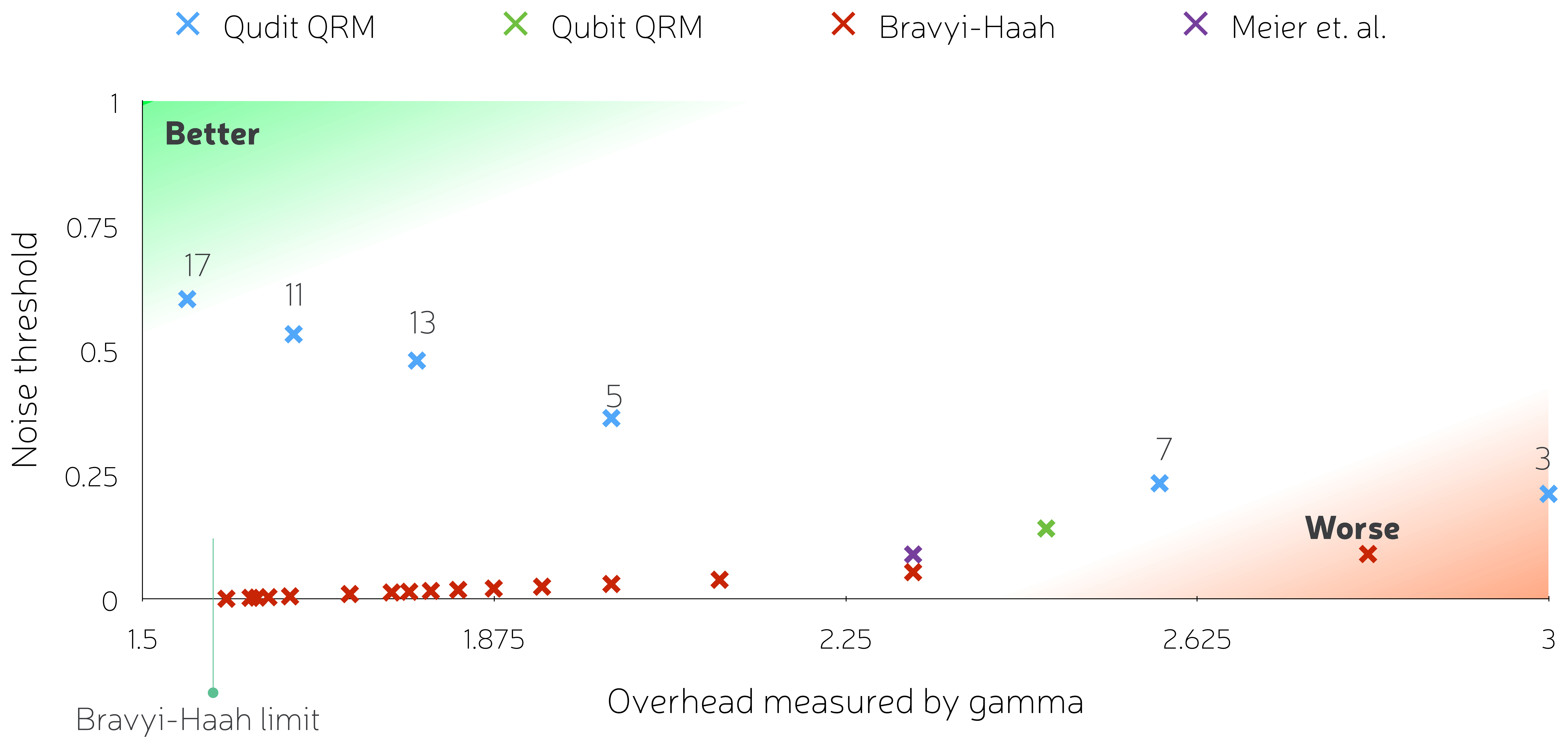
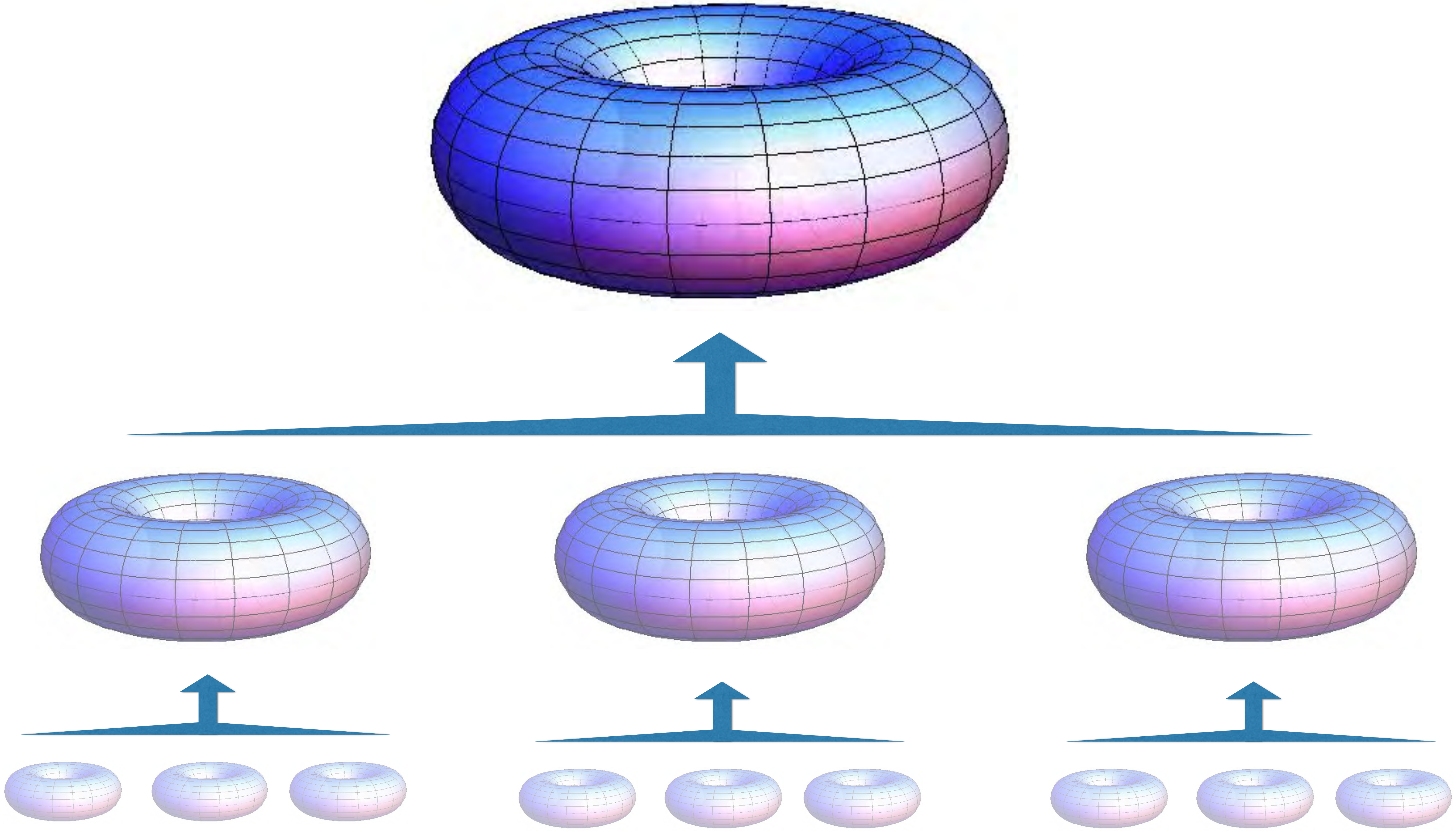
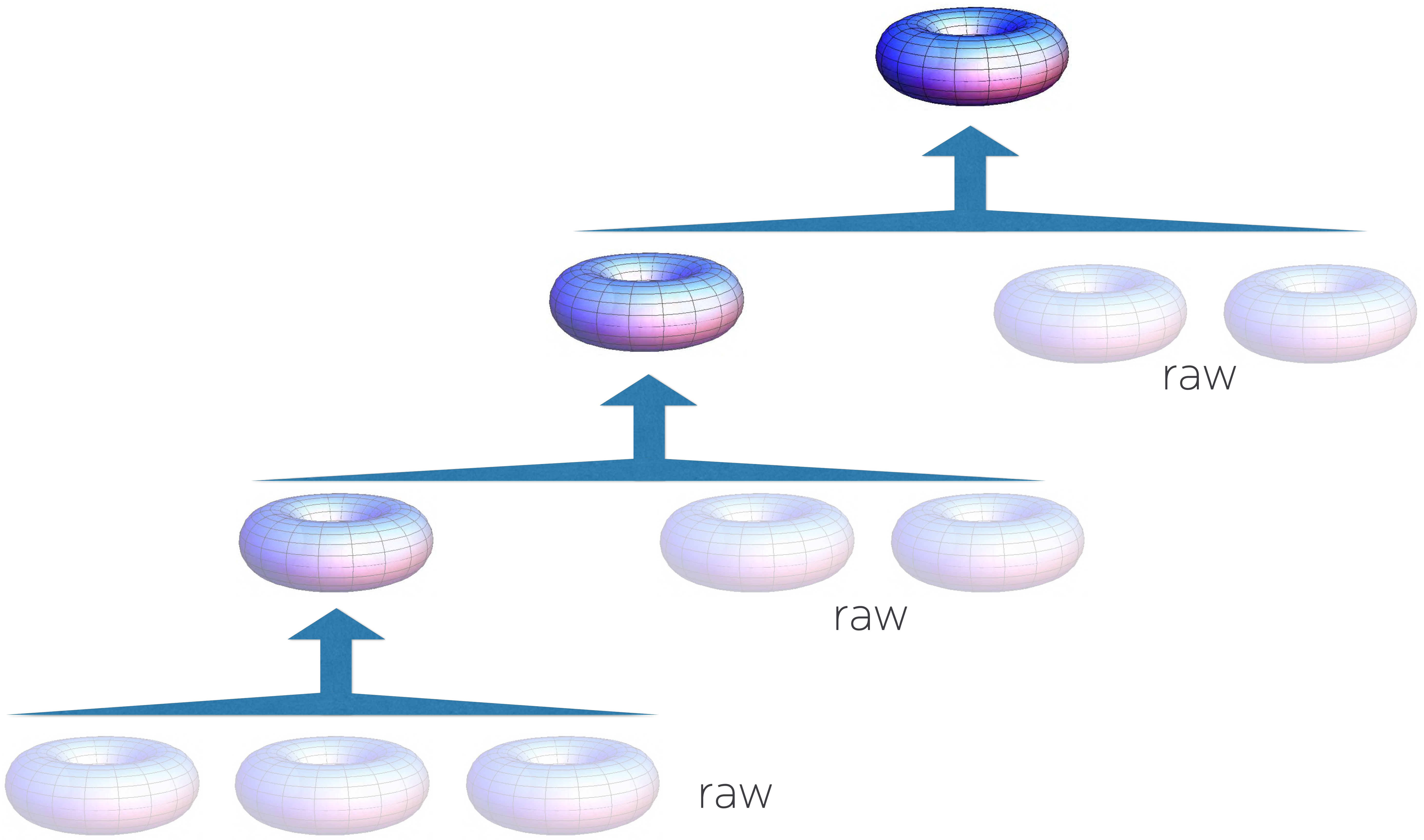


S. Brayvi and J. Haah., Phys. Rev. A **86**, 052329 (2012)
A. Meier, B. Eastin, and E. Knill., QIC **13**, 195 (2013)

BALANCED INVESTMENT



MULTI-LEVEL DISTILLATION



C. Jones., *Phys. Rev. A* **87**, 042305 (2013)

CIRCUIT COMPLEXITY

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Campbell

Value?

Thanks to
D. Browne, H. Anwar, M. Howard,
F. Watson, I. Bengtsson, K. Blanchfield,
and

THANK YOU!

Stooge slide

1. *Hey Earl, can you show that M is in the 3 level of Clifford hierarchy?*
2. *Can you say more about how triorthgonality entails transversality of a non-Clifford?*
3. *Why does the distance increase for higher degree Reed-Muller code?*
4. *Hang on, what is a Reed-Muller codes?*
5. *Can I see more plots please?*
6. *OK, Gamma is high and thresholds are high, but whats about yields?*
7. *Aren't colour codes fun, what about gauge colour codes?*

Background image:
Computation Cloud
installation piece
Libby Heany

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Campbell

1. You want me to show that M is in the 3 level of Clifford hierarchy?

$$M\mathcal{P}M^\dagger = \mathcal{C}$$

Qudits (p>3)

$$M|n\rangle = \omega^{n^3}|n\rangle$$

with $\omega = \exp\left[i\frac{2\pi}{p}\right]$

is diagonal in Z basis so

$$MZM^\dagger = Z$$

$$\begin{aligned} MXM^\dagger|n\rangle &= \omega^{-n^3}MX|n\rangle \\ &= \omega^{-n^3}M|n \oplus 1\rangle \\ &= \omega^{-n^3+(n\oplus 1)^3}|n \oplus 1\rangle \end{aligned}$$

But in exponent all arithmetic is modulo ***p***, so

$$\begin{aligned} MXM^\dagger|n\rangle &= \omega^{-n^3+(n+1)^3}|n \oplus 1\rangle \\ &= \omega^{3n^2+3n+1}|n \oplus 1\rangle \\ &= X\omega^{3n^2+3n+1}|n\rangle \\ &= \omega X\mathcal{Z}_{3,3}|n\rangle \end{aligned}$$

so

$$MXM^\dagger|n\rangle = \omega X\mathcal{Z}_{3,3} \in \mathcal{C}$$

Earl
Campbell

2. You want me to show that triorthgonality entails M transversality?

Qudits (p>3)

$$M|n\rangle = \omega^{n^3}|n\rangle$$

with $\omega = \exp\left[i\frac{2\pi}{p}\right]$

so for a transversal gate

$$M_L = M^{\mu} \otimes M^{\mu} \otimes \ldots M_{\mu}$$

we demand that

$$M_L|m_L\rangle = \omega^{m^3}|m_L\rangle$$

for a flavour of the proof let us look at just

$$M_L|0_L\rangle = |0_L\rangle$$

$$|0_L\rangle = \frac{1}{\sqrt{p}} \sum_{u \in \text{span}[G_0]} |u\rangle$$

so clearly $M_L|u\rangle = |u\rangle, \forall u \in \text{span}[\mathrm{G}_0] = |u\rangle \implies M_L|0_L\rangle = |0_L\rangle$

$$M_L|u\rangle = \bigotimes_n M^{\mu}|u\rangle = \omega^{\mu \sum_j u_j^3}$$

require $M_L|u\rangle = \bigotimes_n M^{\mu}|u\rangle = \omega^{\mu \sum_j u_j^3}|u\rangle$ entails $\mu \sum_j u_j^3 = 0 \pmod{p}$

we write $u = \sum_{v \in \mathcal{U}} v$ where $\mathcal{U}$ is some subset of rows from G_0

$$\begin{aligned} \textstyle\sum_j u_j^3 &= \textstyle\sum_j (\sum_{v \in \mathcal{U}} v_j)^3 \\ &= \textstyle\sum_{v \in \mathcal{U}} \sum_{v' \in \mathcal{U}} \left(\sum_{v'' \in \mathcal{U}} \sum_j v_j v'_j v''_j \right) \end{aligned}$$

zero for triorthgonal matrix

Earl
Campbell

3. You want me to show that code distance scales with r?

Want to find minimum weight $Z[u]$ such that $[Z[u], X[v]] = 0$ for all $v \in G_0$

$[Z[u], X[v]] = \omega^{\sum_j u_j v_j}$ we write $v = \{f(1), f(2), \ldots f(p-1)\},\ u = \{h(1), h(2), \ldots h(p-1)\}$

$$[Z[u], X[v]] = 0 \iff \sum_x f(x)h(x) = 0 \pmod{p}$$

let us assume for brevity $h(x) = x^m$

therefore require $\sum_x x^{m+t} = 0 \pmod{p}$ for all $t \leq r$

Recall: for prime numbers we know

$$\sum_x x^q \pmod{p} = \begin{cases} 0 & q \neq 0 \pmod{p-1} \\ p-1 & q = 0 \pmod{p-1} \end{cases}$$

so,... $$m \leq p - 1 - r$$

but low degree functions can only have a small number of zeros, which tells us $\mathrm{wt}[Z(u)] \geq p - 1 - m$

Therefore $\mathrm{wt}[Z(u)] \geq r$

Earl Campbell

4. Hang on, what is a quantum Reed-Muller code again?

for order ***m*** we have a $[[p^m - 1, 1, d]]$ code

consider functions $f : \mathbb{F}_p^m / \{0\} \rightarrow \mathbb{F}_p$

then G_0 has rows corresponding to different functions and columns evaluate the function at different values

so example if p=2 and m=3, with 3 different function f1, f2, f3

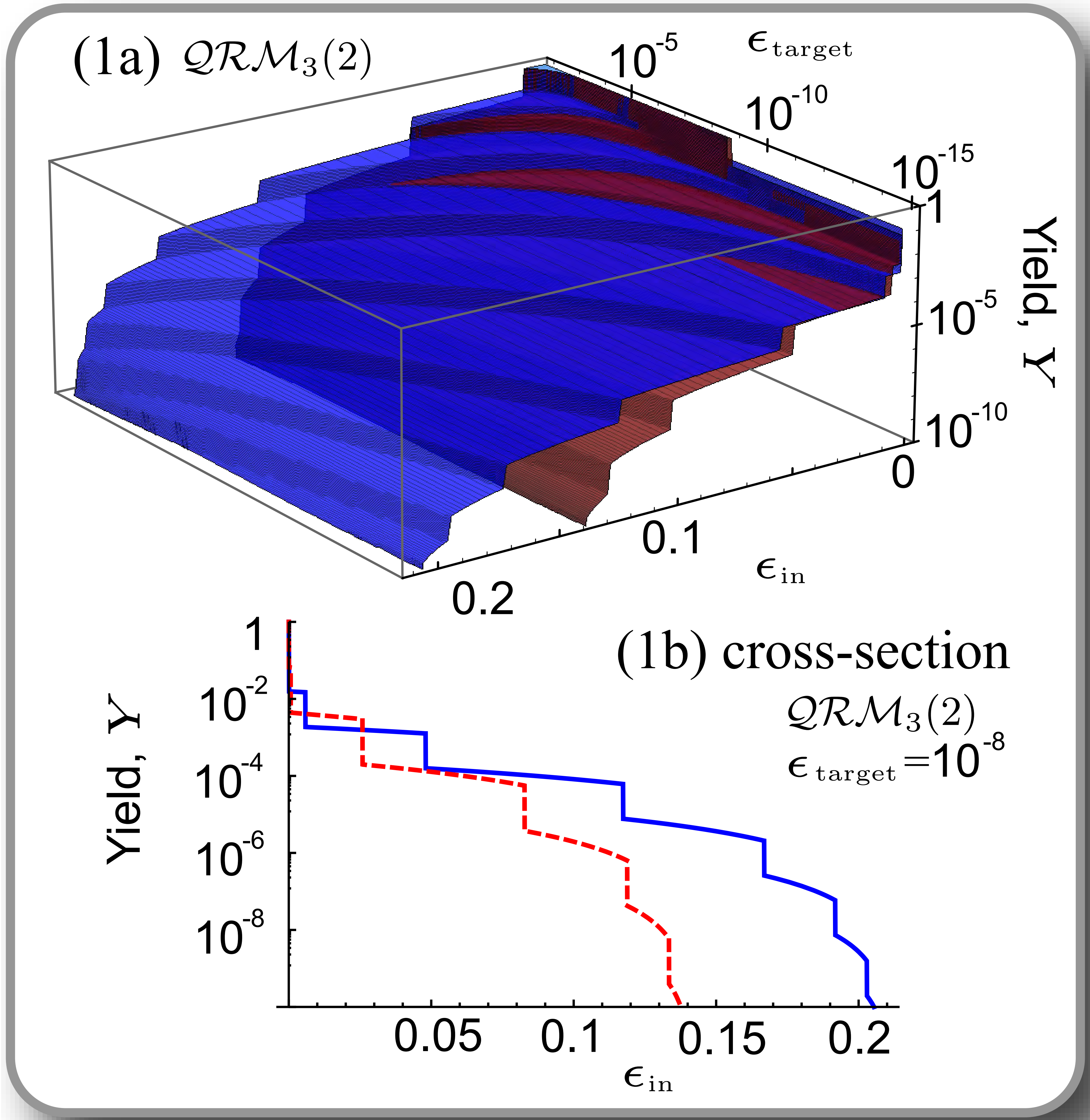
$$G_0 = \begin{pmatrix} f_1(0,0,1) & f_1(0,1,0) & f_1(1,0,0) & f_1(1,1,0) & f_1(1,0,1) & f_1(0,1,1) & f_1(1,1,1) \\ f_2(0,0,1) & f_2(0,1,0) & f_2(1,0,0) & f_2(1,1,0) & f_2(1,0,1) & f_2(0,1,1) & f_2(1,1,1) \\ f_3(0,0,1) & f_3(0,1,0) & f_3(1,0,0) & f_3(1,1,0) & f_3(1,0,1) & f_3(0,1,1) & f_3(1,1,1) \end{pmatrix}$$

these functions can again be polynomials of x1, x2, x3,

For example
$$\begin{array}{rcl} f_1(x_1,x_2,x_3) & = & x_1 \\ f_2(x_1,x_2,x_3) & = & x_2 \\ f_3(x_1,x_2,x_3) & = & x_3 \end{array}$$
gives
$$G_0 = \begin{pmatrix} 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}$$

5/6. More plots please, yes of course….

dimension 3



dimension 5