The advantages of qudit fault-tolerance

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Quantum Error Correction 2014 Zurich

http://earltcampbell.com/research/
WHY QUDITS?


Smith et al.
Anderson et al.
Setun
“Perhaps the prettiest number system of all is the balanced ternary notation”
Donald Knuth
A contemporary approach to fault-tolerant quantum computing

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<td>Use QEC code to reduce noise.</td>
<td>Prepare magic states and inject extra gates.</td>
<td>Compose finite gate set to produce any gate.</td>
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<td>€ £ $</td>
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OVERVIEW
Earl Campbell

**Storage**

**Magic States**

**CODE 1**
Efficient, high-threshold, allows fault-tolerant implementation of (most) Clifford group gates.
*e.g. Toric code*

**CODE 2**
Allows fault-tolerant implementation of a *non-Clifford gate*, *e.g. pi/8 gate*
*e.g. Reed-Muller codes*

fault-tolerant ~ transversal
\[ M_L \sim M \otimes M \otimes M \ldots \]
SUBSYSTEM CODES + GAUGE FIXING

Only 1 code. Potentially fewer resources needed.
But must also allow fault-tolerant non-Clifford gate.

Storage-Magic States

e.g. gauge colour codes, or gauge variants of Reed-Muller codes

Many other alternative, but all rely on these exotic codes

OVERVIEW

Qudit Toric Code  
16 063038 (2014)

Qudit Magic  
Phys. Rev. X  
2 041021 (2012)

Phys. Rev. Lett  
113 230501 (2014)

Benjamin Brown  
Dan Browne  
Hussain Anwar
INTRODUCING QU DITS

Define a basis $\{0, 1, \ldots, p - 1\}$ with all arithmetic modulo $p$.

Imagine states as notches on a clock face.

Pauli-group $\mathcal{P}$:

generators

$X|n\rangle = |n + 1\rangle$

$Z|n\rangle = \omega^n|n\rangle$

where $\omega = e^{i2\pi/p}$

Clifford group $\mathcal{C}$ is normaliser of $\mathcal{P}$ so $\mathcal{C}\mathcal{P}\mathcal{C}^\dagger = \mathcal{P}$

Overcomplete set of generators

$X_{\alpha,\beta}|n\rangle = |\alpha n + \beta\rangle$

$Z_{\alpha,\beta}|n\rangle = \omega^{\alpha n + \beta n^2}|n\rangle$

$H|n\rangle = \frac{1}{\sqrt{p}} \sum_m \omega^{nm}|m\rangle$

$CX|n\rangle|m\rangle = |n\rangle|m + n\rangle$

Assuming $p$ is an odd prime!
The toric code straightforwardly generalizes though we have to adjust the code slightly. Stabilizers are still 4-body terms but “daggerized.”
The toric code straightforwardly generalizes though we have to adjust the code slightly.

Logical operators are still closed loops (in the homological sense).
The toric code straightforwardly generalizes though we have to adjust the code slightly.

Logical operators are still closed loops (in the homological sense)

the shortest loop is length in a code of $2L^2$ qudits.

They are $[[2L^2, 2, L]]$ codes.
Earl Campbell

G. Duclos-Cianci, D. Poulin.,
Phys. Rev. Lett. 104, 050504

S. Bravyi, J. Haah
Phys. Rev. Lett. 111, 200501

MINIMUM WEIGHT PM

BROOM

RENORMALISATION
Decoding and Thresholds

- Thresholds improve with $p$;
- Thresholds follow $\sim 69\%$ of the hashing bound threshold;
- We need to tweak broom to get this.

- Similar results in:

- Advantage persists in when accessing for noisy stabiliser measurement. Thresholds up to 8%
  - F. Watson, H. Anwar, D. Browne  *arXiv*:1411.3028
Magic States Pt. 1
Implementing non-Clifford gates
Good examples of non-Clifford gates

**Qubit**

\[ U|n\rangle = \nu^n |n\rangle \]

with \( \nu = \exp \left[ i \frac{2\pi}{2^3} \right] \)

**Qutrit (p=3)**

\[ V|n\rangle = \tau^n |n\rangle \]

with \( \tau = \exp \left[ i \frac{2\pi}{3^2} \right] \)

**Qudits (p>3)**

\[ M|n\rangle = \omega^{n^3} |n\rangle \]

with \( \omega = \exp \left[ i \frac{2\pi}{p} \right] \)

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Good examples of non-Clifford gates

**Qubit**

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\[ V|n\rangle = \tau^n |n\rangle \]

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**Qudits (p>3) are CUBIC gates**

\[ M|n\rangle = \omega^{n^3} |n\rangle \]

with \( \omega = \exp \left[ i \frac{2\pi}{p} \right] \)

### Clifford hierarchy 3rd level

Conjugates Pauli group to Clifford group

\[ MP\ M^\dagger = C \]

enables gate-injection using resource of magic-states

\[ |M\rangle = M|+\rangle = \frac{1}{\sqrt{p}} \sum_n \omega^{n^3} |n\rangle \]

where

\[ C = MX^m M^\dagger \]
MAGIC STATE DISTILLATION

Given an $[[n,k,d]]$ code with transversal $M$

for noise rate $\epsilon = 1 - \langle M|\rho|M \rangle$ we find

- A threshold: if $\epsilon < \epsilon^*$ then $\epsilon_{\text{out}} < \epsilon$
- Exponential: $\epsilon_{\text{out}} < C\epsilon^d$
- Overhead: $N \leq A_\epsilon \log^\gamma(\epsilon_{\text{target}})$

where we use $N$ raw copies, and

$$\gamma = \log(n/k)/\log(d)$$

---

Define a matrix $G = \left( \begin{matrix} G_0 \\ G_1 \end{matrix} \right)$

Logical states $|\tilde{0}_L\rangle = \frac{1}{\sqrt{p}} \sum_{u \in \text{span}(G_0)} |u\rangle$

$|\tilde{m}_L\rangle = \frac{1}{\sqrt{p}} \sum_{u \in \text{span}(G_0)} |u + G_1 \tilde{m}\rangle$

Define $X[\tilde{v}] = X^{v_1} \otimes X^{v_2} \otimes \ldots \otimes X^{v_n}$

stabilizer generators $X[\tilde{v}]$ for all $\tilde{v} \in G_0$

logical operators $X[\tilde{v}]$ for all $\tilde{v} \in G_1$

**Definition**

A matrix/code is said to be **triorthogonal** if for all $v, v', v'' \in G$

whenever $v = v' = v''$

$$\sum_j v_j^3 = \begin{cases} 0 \quad (\text{mod } p) & , v \in G_0 \\ a \neq 0 \quad (\text{mod } p) & , v \in G_1 \end{cases}$$

otherwise, $\sum_j v_j v_j' v_j'' = 0 \quad (\text{mod } p)$

**Theorem**

A triorthogonal code has a transversal non-Clifford $U, V$ or $M$ (upto a Clifford)


qudits $\square$ unpublished / in preparation
Triorthogonal codes

Reed-Muller codes are \([p^m - 1, 1, ?]\) codes where \(m\) is the order. Simplest qubit triorthogonal code in the “Reed-Muller” family is order 4.

\[
\begin{pmatrix}
 v' \\
v''
\end{pmatrix} = \begin{pmatrix}
 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\
 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{pmatrix}
\]

\[
U |n\rangle = v^n |n\rangle
\]

\[
\nu = \exp \left[ i \frac{2\pi}{2^3} \right]
\]

\[
v \cdot v' \cdot v'' = (110000000000000)
\]

\[
\sum_j v_j \cdot v'_j \cdot v''_j = 2 = 0 \pmod{2}
\]

Definition

A matrix/code is said to be triorthogonal if for all \(v, v', v'' \in G\)

\[
\sum_j v_j^3 = \begin{cases}
 0 \pmod{p}, & v \in G_0 \\
 a \neq 0 \pmod{p}, & v \in G_1
\end{cases}
\]

otherwise, \(\sum_j v_j v'_j v''_j = 0 \pmod{p}\)
Triorthogonal codes
Finding the exotic codes

Simplest qudit “Reed-Muller” code
for dimension 3

\[
\begin{pmatrix}
0 & 0 & 1 & 1 & 1 & 2 & 2 & 2 \\
1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{pmatrix}
\]

\[V |n\rangle = \tau^n |n\rangle\]
with \(\tau = \exp\left[\frac{2\pi i}{3^2}\right]\)

an \([8, 1, 2]\) code so order 2

Compared to 15 qubit:
1. smaller code;
2. worse distance;
3. worse overhead (gamma);
4. threshold?
In higher dimensions, 
$$\epsilon = 1 - \langle M | \rho | M \rangle$$
doesn’t uniquely identify the state.

Let us look a slice of state space
(which we can always project onto)
We parameterize by 
$$z = \text{tr}[M \rho]$$

---

Triorthogonal codes
Finding the exotic codes

Simplest qudit “Reed-Muller” code
for dimension 5
\[
\begin{pmatrix}
1 & 2 & 3 & 4 \\
1 & 1 & 1 & 1
\end{pmatrix}
\]

for dimension 7
\[
\begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 \\
1 & 1 & 1 & 1 & 1 & 1
\end{pmatrix}
\]

for dimension 11
\[
\begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{pmatrix}
\]

Gives a triorthonormal family of order 1 Reed-Muller codes.
[[p − 1, 1, 2]]

M \ket{n} = \omega^n \ket{n}
with \( \omega = \exp \left(i \frac{2\pi}{p} \right) \)
Threshold

Threshold \( \epsilon_{\text{dep}} \)

Qudit dimension p

Overhead

Gamma

Qudit dimension p

\[ N \leq A_\epsilon \log^\gamma(\epsilon_{\text{target}}) \]
Magic States pt 2
Beyond linear functions
Reed-Muller codes in more detail

- stick to first-order codes \([(p^m - 1), 1, ?]\)
- define generating matrix in terms of functions

\[
G = \left( \frac{G_0}{G_1} \right) = \begin{pmatrix}
    f_1(1) & f_1(2) & \cdots & f_1(p - 1) \\
    f_2(1) & f_2(2) & \cdots & f_2(p - 1) \\
    \vdots & \vdots & \ddots & \vdots \\
    f_r(1) & f_r(2) & \cdots & f_r(p - 1) \\
    1 & 1 & \cdots & 1
\end{pmatrix}
\]

- earlier we had only 1 function \(f_1(x) = x\)
- a degree-\(r\) Reed-Muller code to has generating functions

\[
f_1(x) = x, \ f_2(x) = x^2, \ldots \ f_r(x) = x^r
\]
Reed-Muller codes in more detail

- A degree-\( r \) Reed-Muller code has generating functions
  \[
  f_1(x) = x, \; f_2(x) = x^2, \; f_r(x) = x^r
  \]

**Result**

A degree-\( r \) Reed-Muller code is triorthogonal if

\[
r < \frac{(p - 1)}{3}
\]

Furthermore, it can detect up-to \( r \) errors.

1. Consider a triple-product from \( G_0 \)

\[
g(x) = f_a(x) f_b(x) f_c(x) = x^q
\]

\[
q = a + b + c
\]

2. For prime numbers we know

\[
\sum x^q \pmod{p} = \begin{cases} 
0 & q \neq 0 \pmod{p-1} \\
p - 1 & q = 0 \pmod{p-1}
\end{cases}
\]

3. Triorthgonality ensured if \( \forall a, b, c \leq r \) we have \( a + b + c < p - 1 \)

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PERFORMANCE

Overhead

Threshold

no cloning limit for QEC
OVERHEAD

I. Bengtsson, K. Blanchfield, E. Campbell, M. Howard


GRAND COMPARISON

- Qudit QRM
- Qubit QRM
- Bravyi-Haah
- Meier et. al.

Noise threshold vs. Overhead measured by gamma

- Better
- Worse

Bravyi-Haah limit

BALANCED INVESTMENT
MULTI-LEVEL DISTILLATION

Value?
THANK YOU!

Thanks to
D. Browne, H. Anwar, M. Howard,
F. Watson, I. Bengtsson, K. Blanchfield,
and
1. Hey Earl, can you show that $M$ is in the 3 level of Clifford hierarchy?
2. Can you say more about how triorthgonality entails transversality of a non-Clifford?
3. Why does the distance increase for higher degree Reed-Muller code?
4. Hang on, what is a Reed-Muller codes?
5. Can I see more plots please?
6. OK, Gamma is high and thresholds are high, but what's about yields?
7. Aren't colour codes fun, what about gauge colour codes?
1. You want me to show that $M$ is in the 3 level of Clifford hierarchy?

$$MPM^\dagger = C$$

Qudits ($p>3$)

$$M|n\rangle = \omega^n |n\rangle$$

with $\omega = \exp\left[\frac{2\pi}{p}\right]$ is diagonal in $Z$ basis so

$$MZM^\dagger = Z$$

$$MXM^\dagger|n\rangle = \omega^{-n^3}MX|n\rangle$$

$$= \omega^{-n^3}M|n \oplus 1\rangle$$

$$= \omega^{-n^3+(n \oplus 1)^3}|n \oplus 1\rangle$$

But in exponent all arithmetic is modulo $p$, so

$$MXM^\dagger |n\rangle = \omega^{-n^3+(n+1)^3}|n \oplus 1\rangle$$

$$= \omega^{3n^2+3n+1}|n \oplus 1\rangle$$

$$= X\omega^{3n^2+3n+1}|n\rangle$$

$$= \omega X\mathbb{Z}_{3,3}|n\rangle$$

so

$$MXM^\dagger|n\rangle = \omega X\mathbb{Z}_{3,3} \in C$$
2. You want me to show that triorthgonality entails $M$ transversality?

Qudits ($p>3$)

$$M|n\rangle = \omega^{n^3}|n\rangle$$

with $\omega = \exp\left[\frac{2\pi i}{p}\right]$  

so for a transversal gate

$$ML = M^\mu \otimes M^\mu \otimes \ldots M^\mu$$

we demand that

$$ML|m_L\rangle = \omega^{m^3}|m_L\rangle$$

for a flavour of the proof let us look at just

$$ML|0_L\rangle = |0_L\rangle$$

$$|0_L\rangle = \frac{1}{\sqrt{p}} \sum_{u \in \text{span}[G_0]} |u\rangle$$

so clearly $ML|u\rangle = |u\rangle, \forall u \in \text{span}[G_0] = |u\rangle \implies ML|0_L\rangle = |0_L\rangle$

$$ML|u\rangle = \bigotimes_n M^\mu|u\rangle = \omega^{\mu} \sum_j u_j^3$$

require $ML|u\rangle = \bigotimes_n M^\mu|u\rangle = \omega^{\mu} \sum_j u_j^3 |u\rangle$ entails $\mu \sum_j u_j^3 = 0 \pmod{p}$

we write $u = \sum_{v \in U} v$ where $U$ is some subset of rows from $G_0$

$$\sum_j u_j^3 = \sum_j (\sum_{v \in U} v_j)^3 = \sum_{v \in U} \sum_{v' \in U} \left(\sum_{v'' \in U} \sum_j v_j v'_j v''_j\right)$$

zero for triorthogonal matrix
3. You want me to show that code distance scales with r?

Want to find minimum weight $Z[u]$ such that $[Z[u], X[v]] = 0$ for all $v \in G_0$

$[Z[u], X[v]] = \omega \sum_j u_j v_j$ we write $v = \{f(1), f(2), \ldots f(p - 1)\}$, $u = \{h(1), h(2), \ldots h(p - 1)\}$

$[Z[u], X[v]] = 0 \iff \sum x f(x) h(x) = 0 \pmod{p}$

let us assume for brevity $h(x) = x^m$

therefore require $\sum x^{m+t} = 0 \pmod{p}$ for all $t \leq r$

Recall: for prime numbers we know

$\sum x^q \pmod{p} = \begin{cases} 0 & q \neq 0 \pmod{p - 1} \\ p - 1 & q = 0 \pmod{p - 1} \end{cases}$

so,... $m \leq p - 1 - r$

but low degree functions can only have a small number of zeros, which tells us $\text{wt}[Z(u)] \geq p - 1 - m$

Therefore $\text{wt}[Z(u)] \geq r$
4. Hang on, what is a quantum Reed-Muller code again?

for order $m$ we have a $[[p^m - 1, 1, d]]$ code

consider functions $f : \mathbb{F}_p^m / \{0\} \rightarrow \mathbb{F}_p$

then $G_0$ has rows corresponding to different functions and columns evaluate the function at different values

so example if $p=2$ and $m=3$, with 3 different function $f_1, f_2, f_3$

$$G_0 = \begin{pmatrix}
    f_1(0, 0, 1) & f_1(0, 1, 0) & f_1(1, 0, 0) & f_1(1, 1, 0) & f_1(1, 0, 1) & f_1(0, 1, 1) & f_1(1, 1, 1) \\
    f_2(0, 0, 1) & f_2(0, 1, 0) & f_2(1, 0, 0) & f_2(1, 1, 0) & f_2(1, 0, 1) & f_2(0, 1, 1) & f_2(1, 1, 1) \\
    f_3(0, 0, 1) & f_3(0, 1, 0) & f_3(1, 0, 0) & f_3(1, 1, 0) & f_3(1, 0, 1) & f_3(0, 1, 1) & f_3(1, 1, 1)
\end{pmatrix}$$

these functions can again be polynomials of $x_1, x_2, x_3,$

For example

$$f_1(x_1, x_2, x_3) = x_1$$

$$f_2(x_1, x_2, x_3) = x_2$$

$$f_3(x_1, x_2, x_3) = x_3$$

Gives

$$G_0 = \begin{pmatrix}
    0 & 0 & 1 & 1 & 1 & 0 & 1 \\
    0 & 1 & 0 & 1 & 0 & 1 & 1 \\
    1 & 1 & 0 & 1 & 0 & 1 & 1
\end{pmatrix}$$
PERFORMANCE OF DIMENSION 5 CASE

5/6. More plots please, yes of course....

dimension 3

(1a) $\mathcal{QRM}_3(2)$

(1b) cross-section

$\mathcal{QRM}_3(2)$

$\varepsilon_{\text{target}} = 10^8$

Yield, $Y$

$\varepsilon_{\text{in}}$

0.1

dimension 5

(2a) $\mathcal{QRM}_5(1)$

(2b) cross-section

$\mathcal{QRM}_5(1)$

$\varepsilon_{\text{target}} = 10^8$

Yield, $Y$

$\varepsilon_{\text{in}}$

0.1