FAULT-TOLERANT LOGICAL GATES IN QUANTUM ERROR-CORRECTING CODES

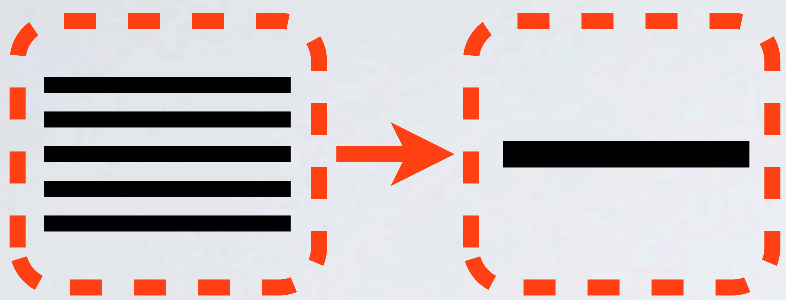
Fernando Pastawski with Beni Yoshida

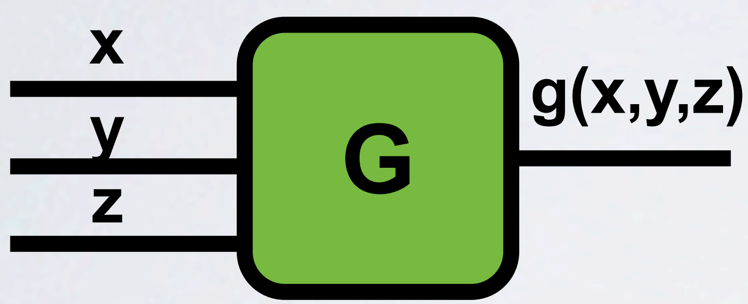
arXiv:1408.1720 (soon PRA)

QEC 2014, Zurich

ROBUST GATES FROM NOISY ONES

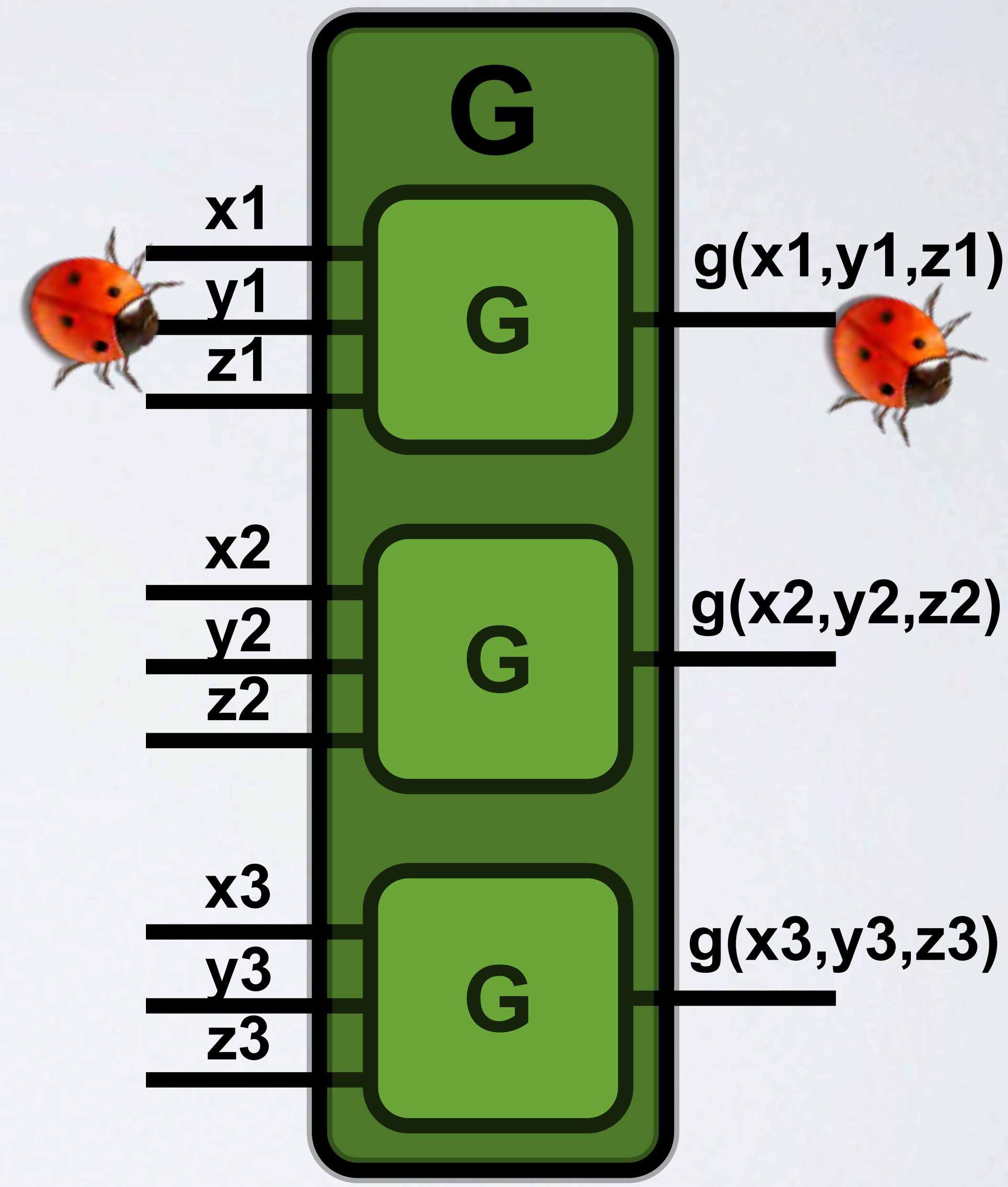
Repetition code





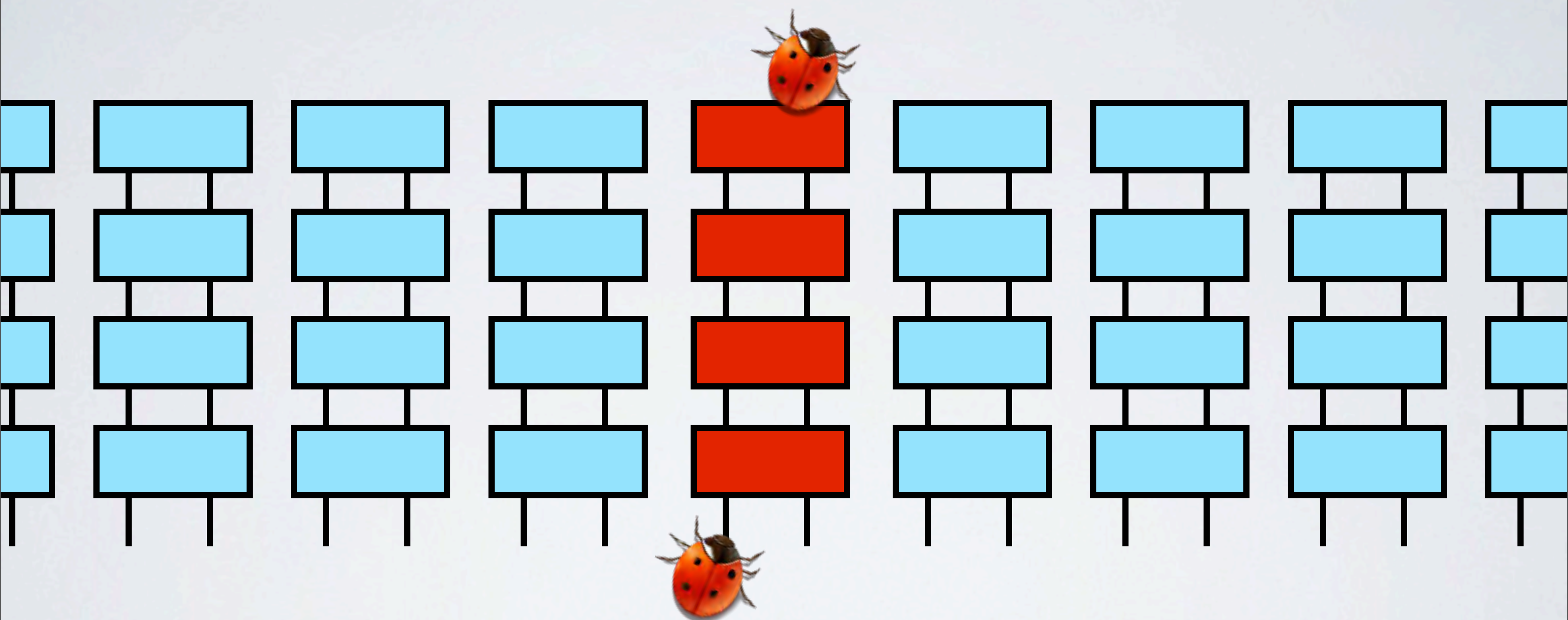
- Transverse gates.
  - Benign error propagation.
- Single errors are recoverable.

Transverse gates



J. von Neumann. In C. Shannon and J. McCarthy (editors) Automata Studies, pages 43--98, Princeton University Press. (1956).

ERROR PROPAGATION IN **TRANSVERSE** CIRCUITS



Errors only propagate within blocks.

Example: Cnot in CSS stabilizer codes.

THE EASTIN & KNILL THEOREM (2008)

- Transversal logical gates are not universal for QC

PRL **102,** 110502 (2009) PHYSICAL REVIEW LETTERS week ending 20 MARCH 2009

Restrictions on Transversal Encoded Quantum Gate Sets

Bryan Eastin* and Emanuel Knill

National Institute of Standards and Technology, Boulder, Colorado 80305, USA

(Received 28 November 2008; published 18 March 2009)

Transversal gates play an important role in the theory of fault-tolerant quantum computation due to their simplicity and robustness to noise. By definition, transversal operators do not couple physical subsystems within the same code block. Consequently, such operators do not spread errors within code blocks and are, therefore, fault tolerant. Nonetheless, other methods of ensuring fault tolerance are required, as it is invariably the case that some encoded gates cannot be implemented transversally. This observation has led to a long-standing conjecture that transversal encoded gate sets cannot be universal. Here we show that the ability of a quantum code to detect an arbitrary error on any single physical subsystem is incompatible with the existence of a universal, transversal encoded gate set for the code.

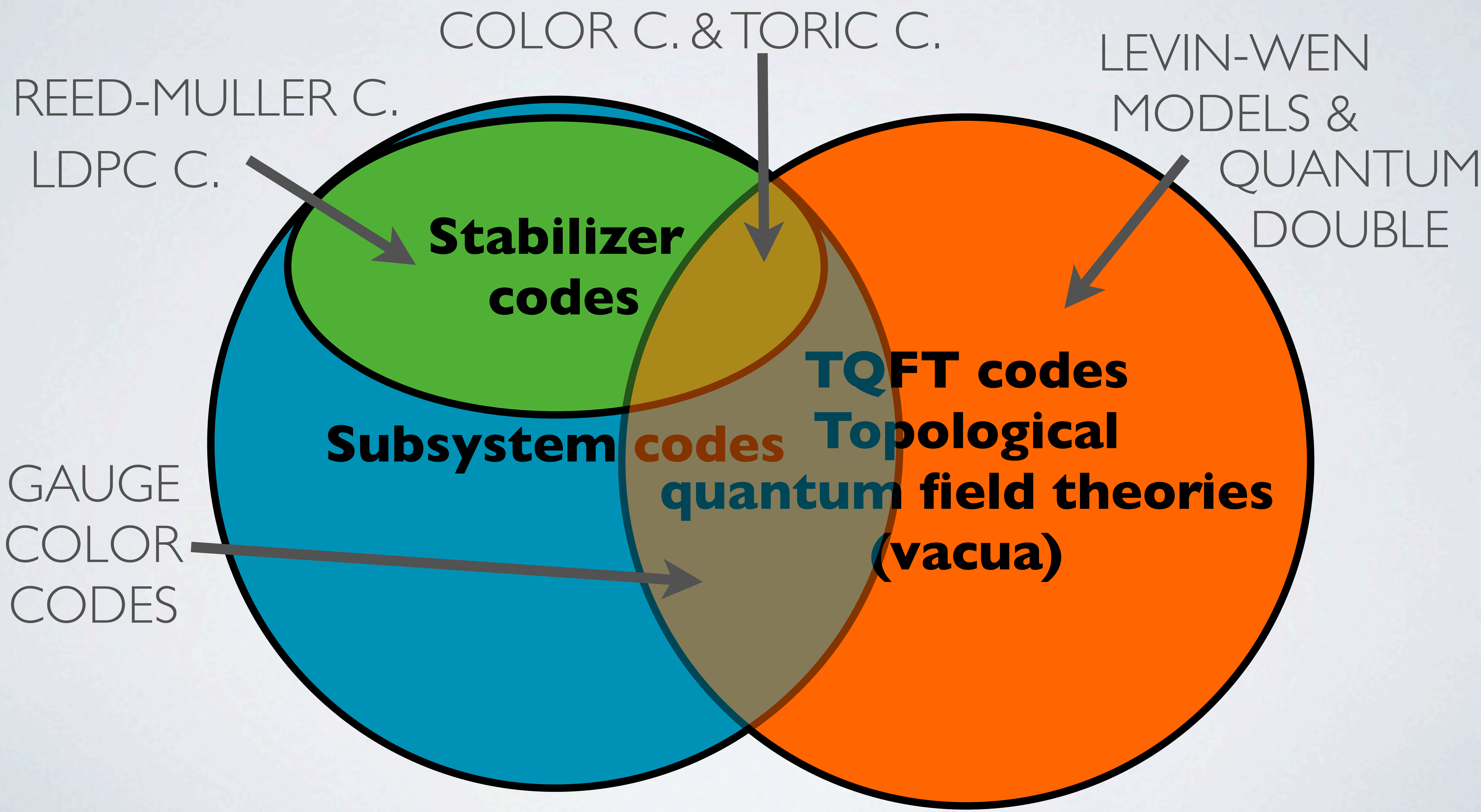
DOI: 10.1103/PhysRevLett.102.110502 PACS numbers: 03.67.Lx, 03.67.Pp

Don't panic ! Fault-tolerant computation is still possible.

B. Eastin, E. Knill, Restrictions on Transversal Encoded Quantum Gate Sets, arXiv: 0811.4262 [quant-ph] (2008).

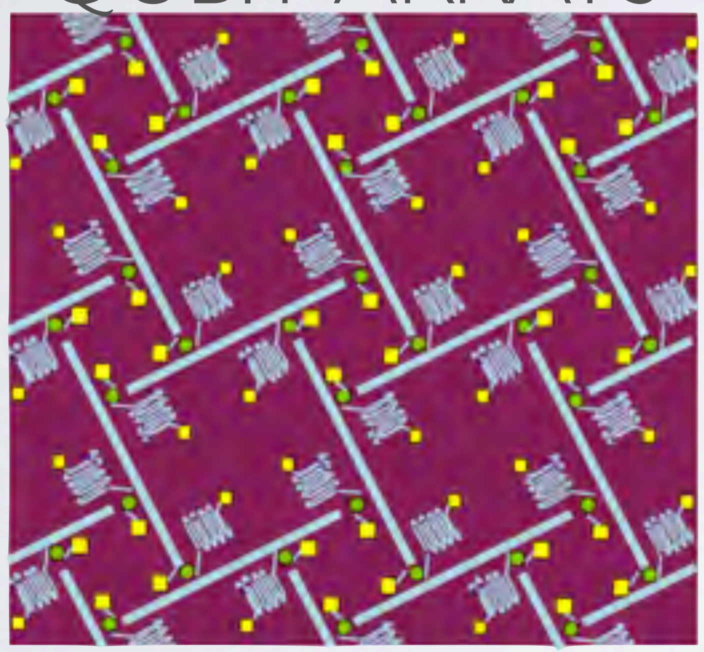
FAMILIES OF QECC

(A LOT ABOUT A LITTLE, A LITTLE ABOUT A LOT)

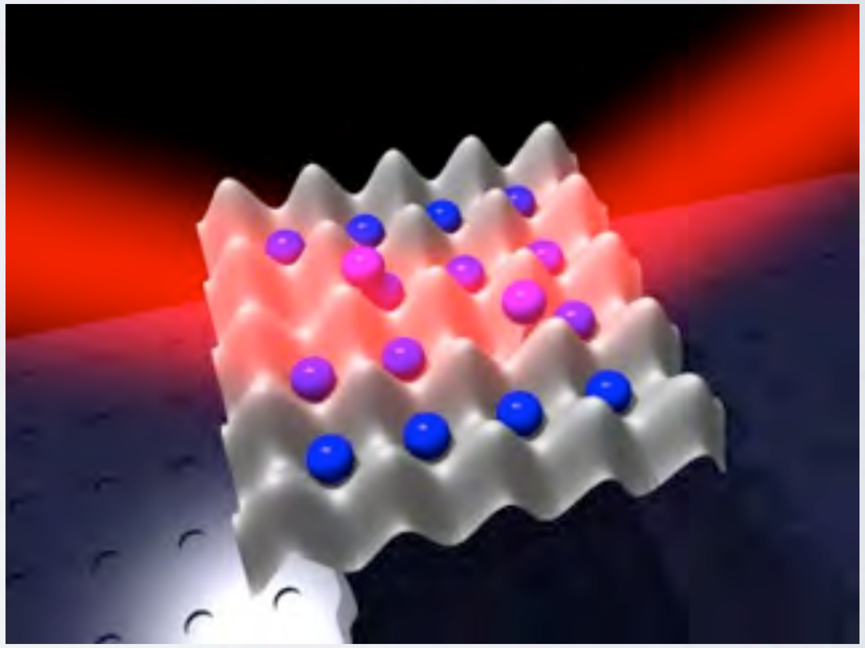


LOGICAL GATES FROM LOCAL INTERACTIONS IN TOPOLOGICAL CODES

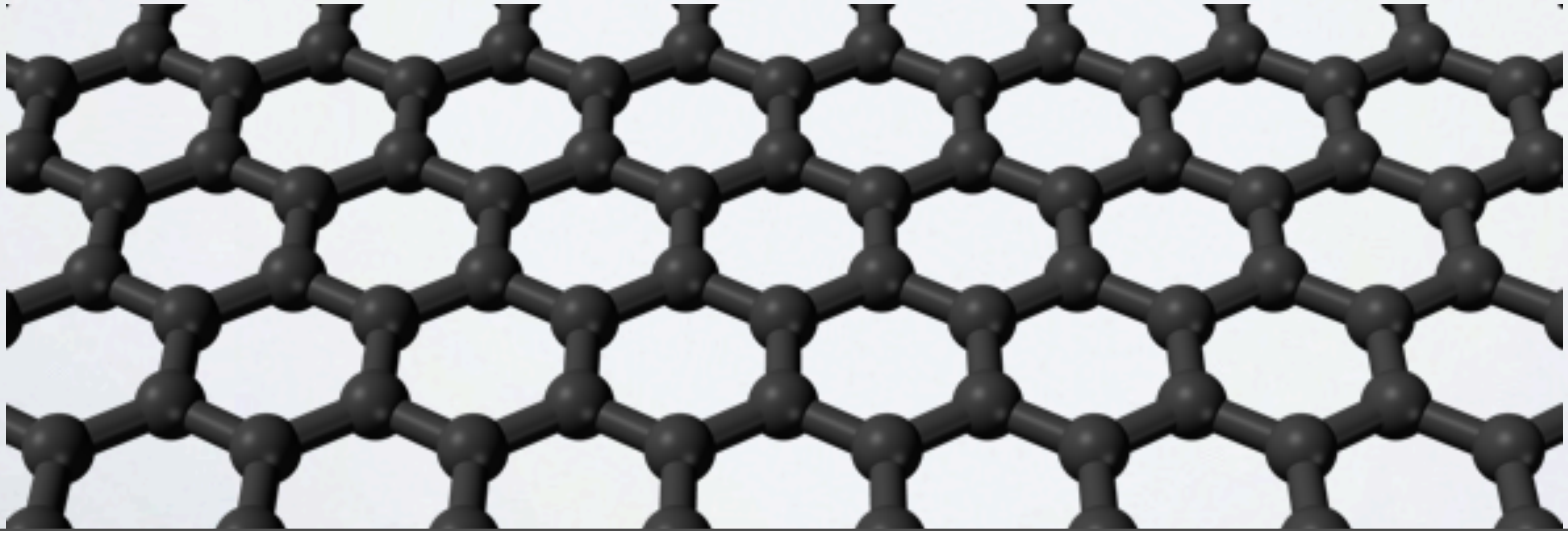
SUPERCONDUCTING QUBIT ARRAYS



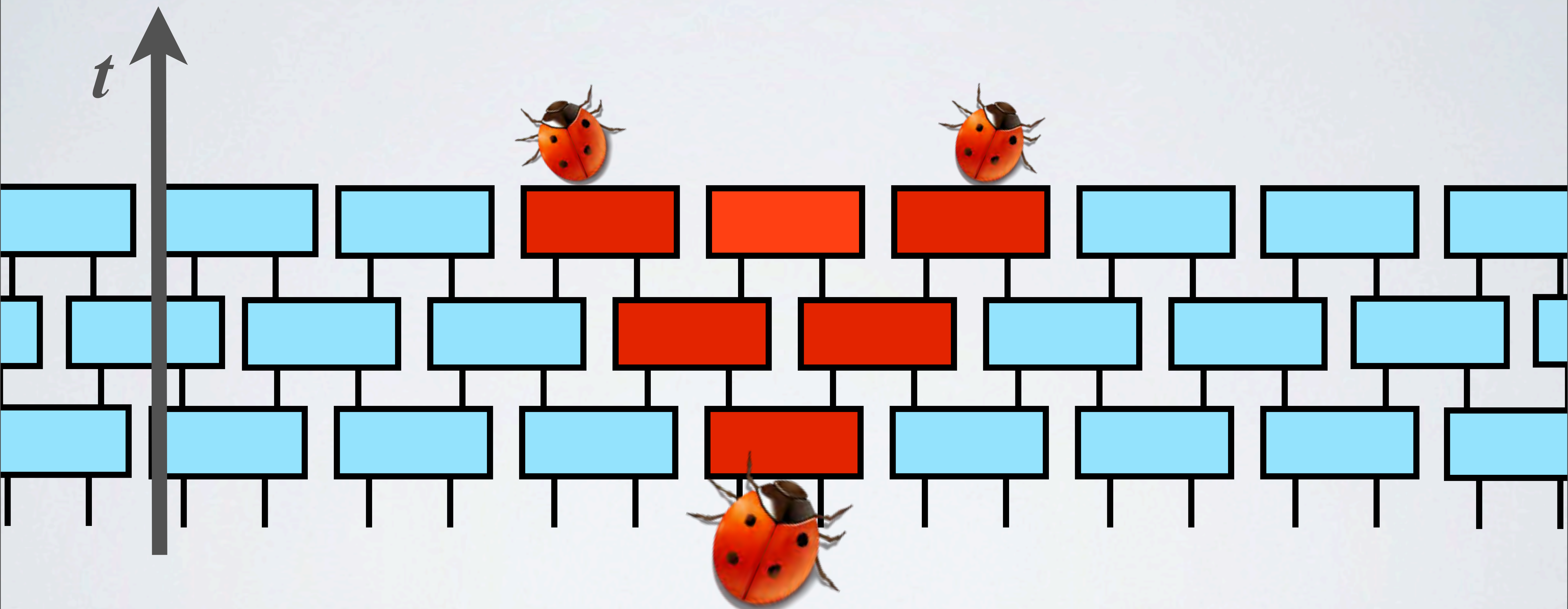
OPTICAL LATICES



SOLID STATE



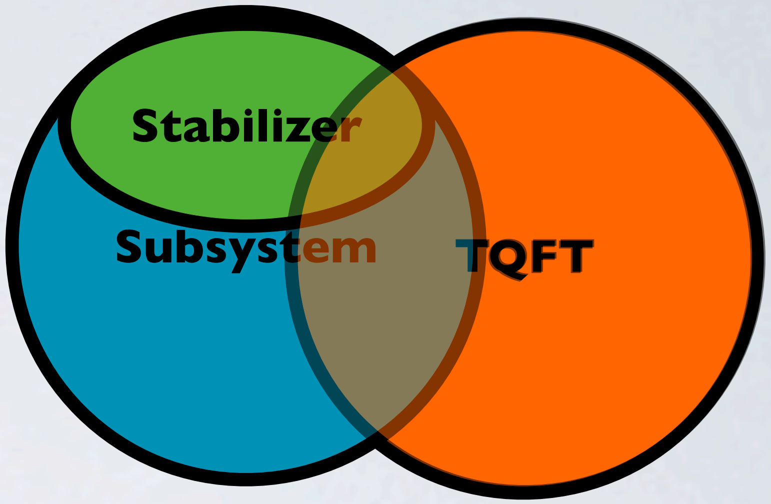
ERROR PROPAGATION IN **LOCAL** CIRCUITS



Errors only propagate geometrically by some constant radius.

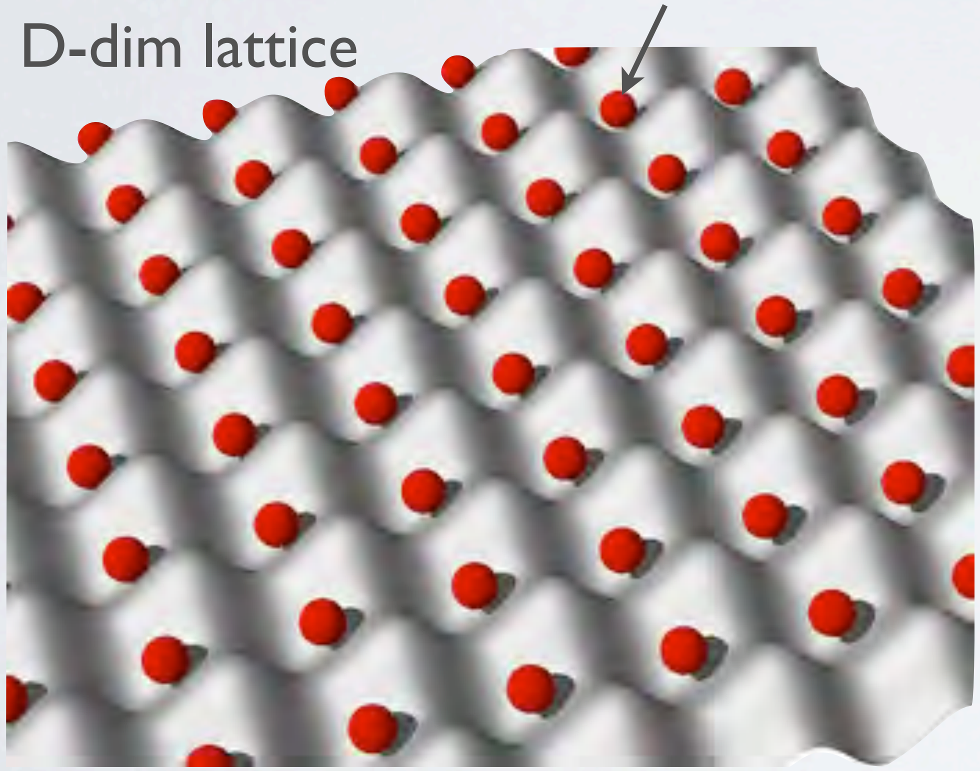
THE BRAVYI-KÖNIG THEOREM (2012)

- Under a more physically realistic setting



Logical gate U : low-depth unitary gate (i.e. Local unitary)

D-dim lattice



Theorem

- For a stabilizer code in D dim, logical gates implementable by local circuits are restricted to the D-th level of the Clifford hierarchy.

Bravyi, S., & König, R. (2013).
Classification of Topologically Protected Gates for Local Stabilizer Codes.
Physical Review Letters, 110(17), 170503.

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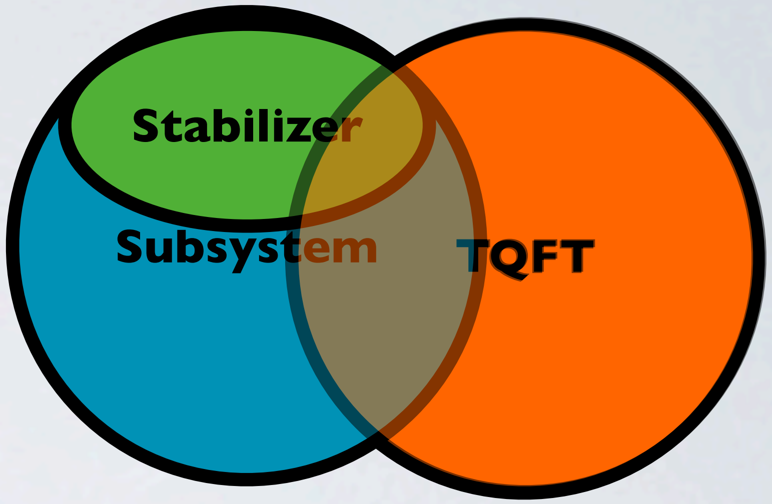

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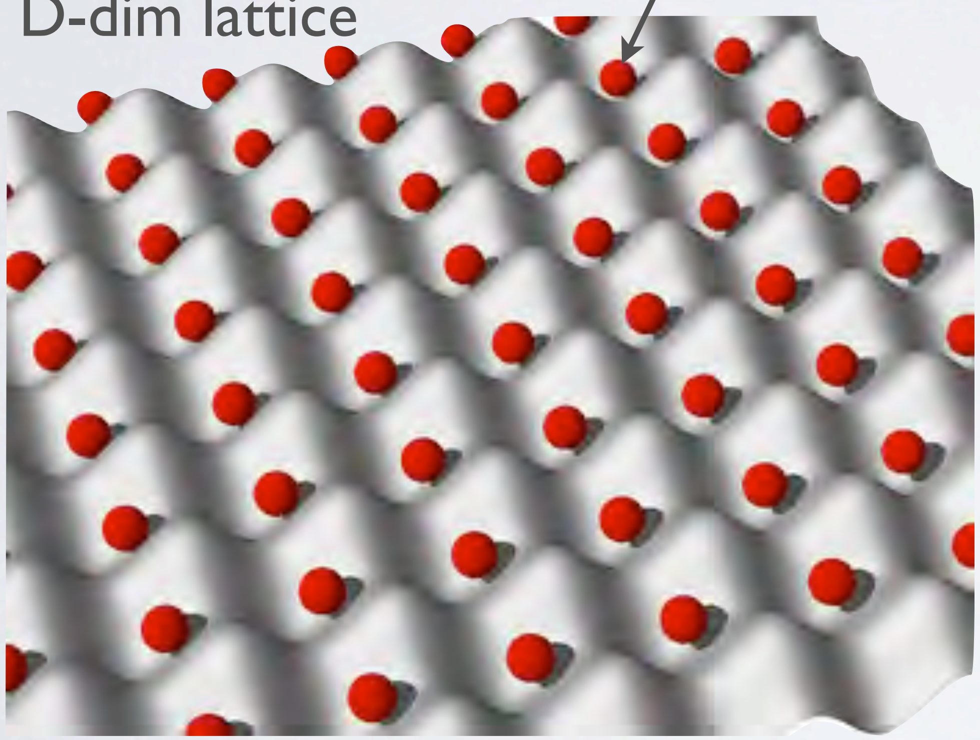

Theorem

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???

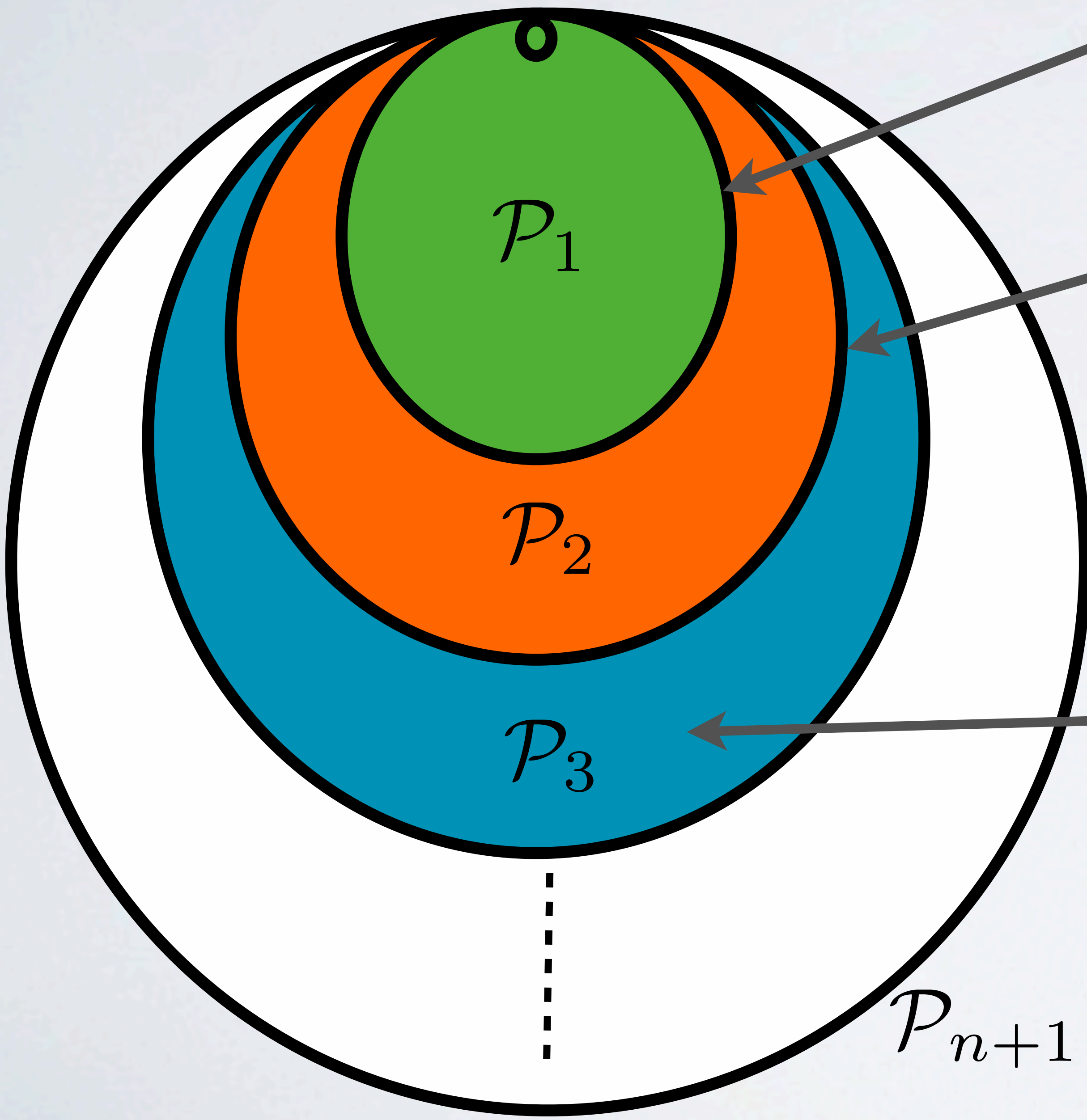
Bravyi, S., & König, R. (2013).
Classification of Topologically Protected Gates for Local Stabilizer Codes.
Physical Review Letters, 110(17), 170503.

CLIFFORD HIERARCHY

Gottesman, D., & Chuang, I. L. (1999).
Demonstrating the viability of universal quantum computation using teleportation and single-qubit operations, 402(6760), 390–393.

Sets of unitary transformations on N qubits

$$\mathcal{P}_0 \equiv \mathbb{C}$$



$\mathcal{P}_1$ = Pauli group: X,Y,Z, XX, -ZZIZZI, ...

$\mathcal{P}_2$ = Clifford group: CNOT, Hadamard, R2,...

$$\mathcal{P}_2 = \mathbb{C}\langle\langle H, Cnot, P\rangle\rangle$$

Clifford group is classically simulable.

$$P = R_2 = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$$

$\mathcal{P}_3$ has R3 & Toffoli. Not a group, but a set.

$$R_3 = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}$$

$$\mathcal{P}_{n+1} = \{U : \forall V \in \text{Pauli}, UVU^\dagger V^\dagger \in \mathcal{P}_n\}$$

D. Gottesman (1998), The Heisenberg Representation of Quantum Computers, arXiv:quant-ph/9807006.

EXAMPLE: COLOR CODE

- Transverse Gates: $\boldsymbol{X} = X^{\otimes N}$ $\boldsymbol{Y} = -Y^{\otimes N}$ $\boldsymbol{Z} = Z^{\otimes N}$

$$\boldsymbol{H} = H^{\otimes N} \qquad \boldsymbol{Cnot} = Cnot^{\otimes N}$$

$$\boldsymbol{P} = \bigotimes_{j\in[1,N]} P_j^{\pm 1}$$

Full transverse Clifford group!

(assuming logical qubits can be stacked)

Bombin, H., & Martin-Delgado, M. (2007). *Topological Computation without Braiding.* Physical Review Letters, 98(16), 160502.

Daniel Nigg, Markus Müller, Esteban A. Martinez, Philipp Schindler, Markus Hennrich, Thomas Monz, Miguel Angel Martin-Delgado, and Rainer Blatt. *Quantum Computations on a Topologically Encoded Qubit.* Science 2014

LITERATURE & RESULTS

Quantum Code type	Transverse gate no geometry	Const. depth circ. + locally defined code
Stabilizer	B. Zeng, A. Cross & I. L. Chuang 2007	S. Bravyi & R. König 2013
Arbitrary	B. Easting & E. knill 2008	Now!
Subsystem	Friday!	F. Pastawski & B. Yoshida 2014
TQFT		M. Beverland, R. T. König, F. Pastawski, J. Preskill & S. Sijher 2014

OUTLINE

- **Cleaning** in Quantum error correcting codes
  - Stabilizer codes
  - Sub-system codes
- Central proof ideas.
- Summary of gate constraints.
- Conclusions & further directions

STABILIZER CODES SUBSYSTEM CODES & CLEANING LEMMAS

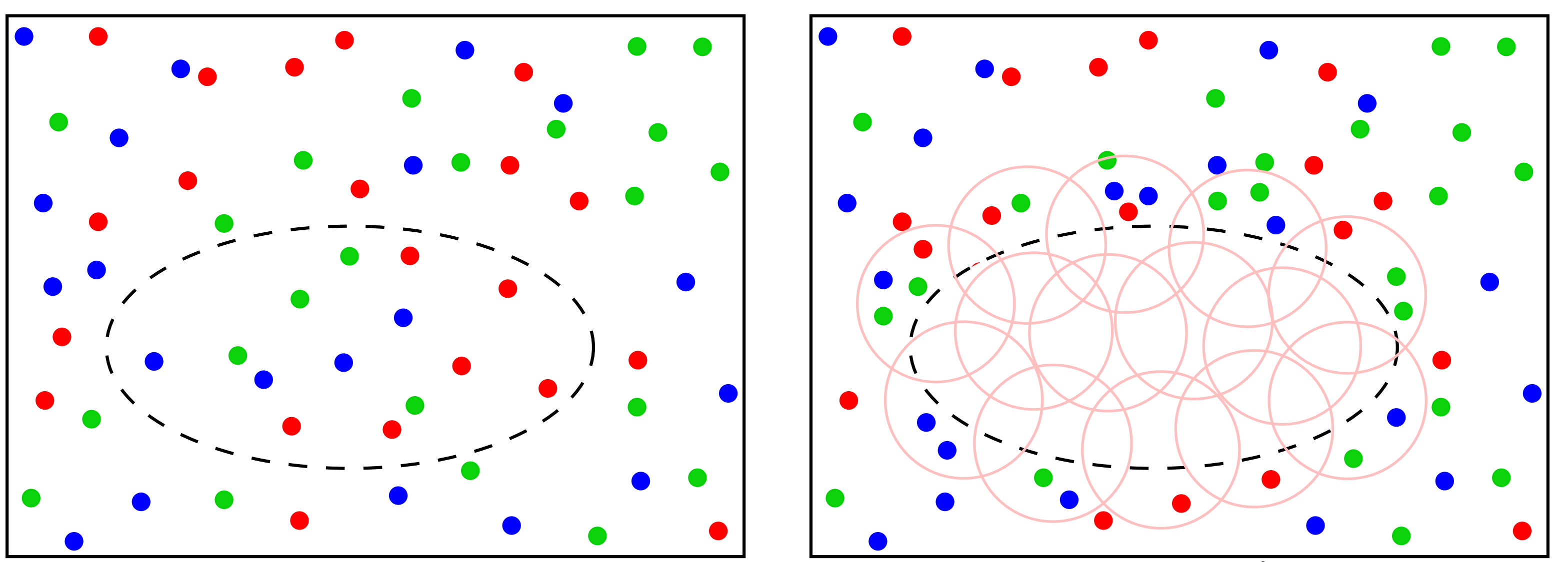
PRE-CLEANING LEMMA

Errors on a region $\boldsymbol{R}$ (subset of qubits) are detectable iff

$$\mathsf{Sup}(U) \subseteq R \quad \Rightarrow \quad P_0 U P_0 = \alpha P_0$$

Correctable regions: $\exists \Lambda_R : \rho = P_0 \rho \quad \Rightarrow \quad \Lambda_R \mathrm{Tr}_R[\rho] = \rho$

can be cleaned. $\mathrm{Tr}[O\rho] = \mathrm{Tr}[\Lambda_R^\dagger(O)\rho]$



$O \qquad\qquad O_{\bar{R}} = \Lambda_R^\dagger(O)$

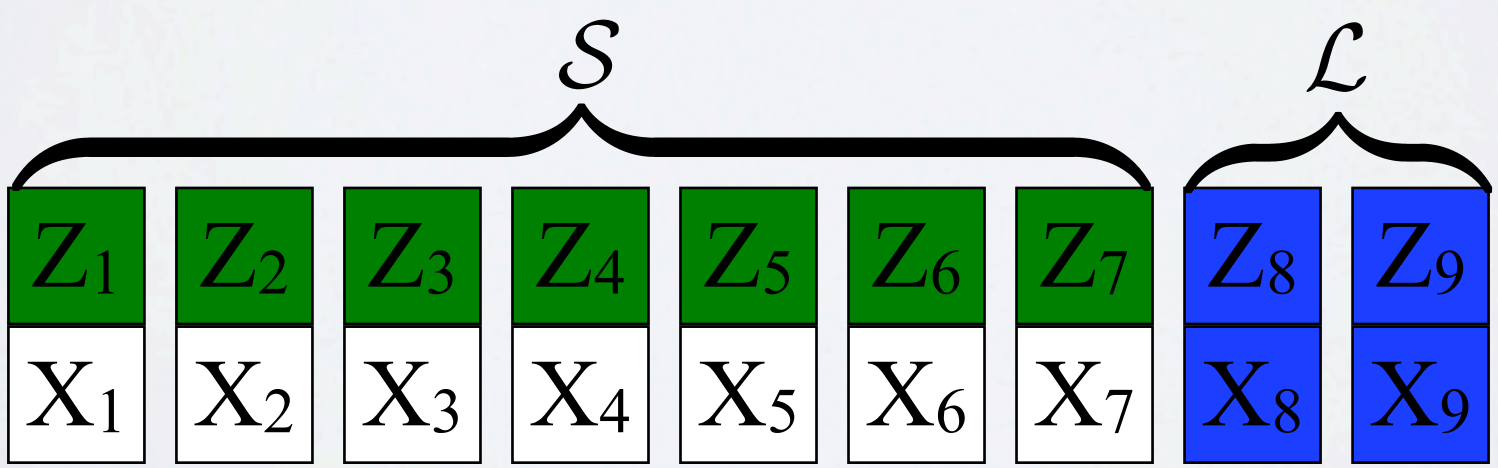
E. Knill, R. Laflamme, Phys. Rev. A 55, 900 (1997).

THE STABILIZER FORMALISM

- The Pauli group $\mathcal{P} = \langle i, X_j, Z_j \rangle$ $\quad |\mathcal{P}| = 4^{(N+1)}$
- A stabilizer subgroup $-1 \notin \mathcal{S} \subset \mathcal{P}$

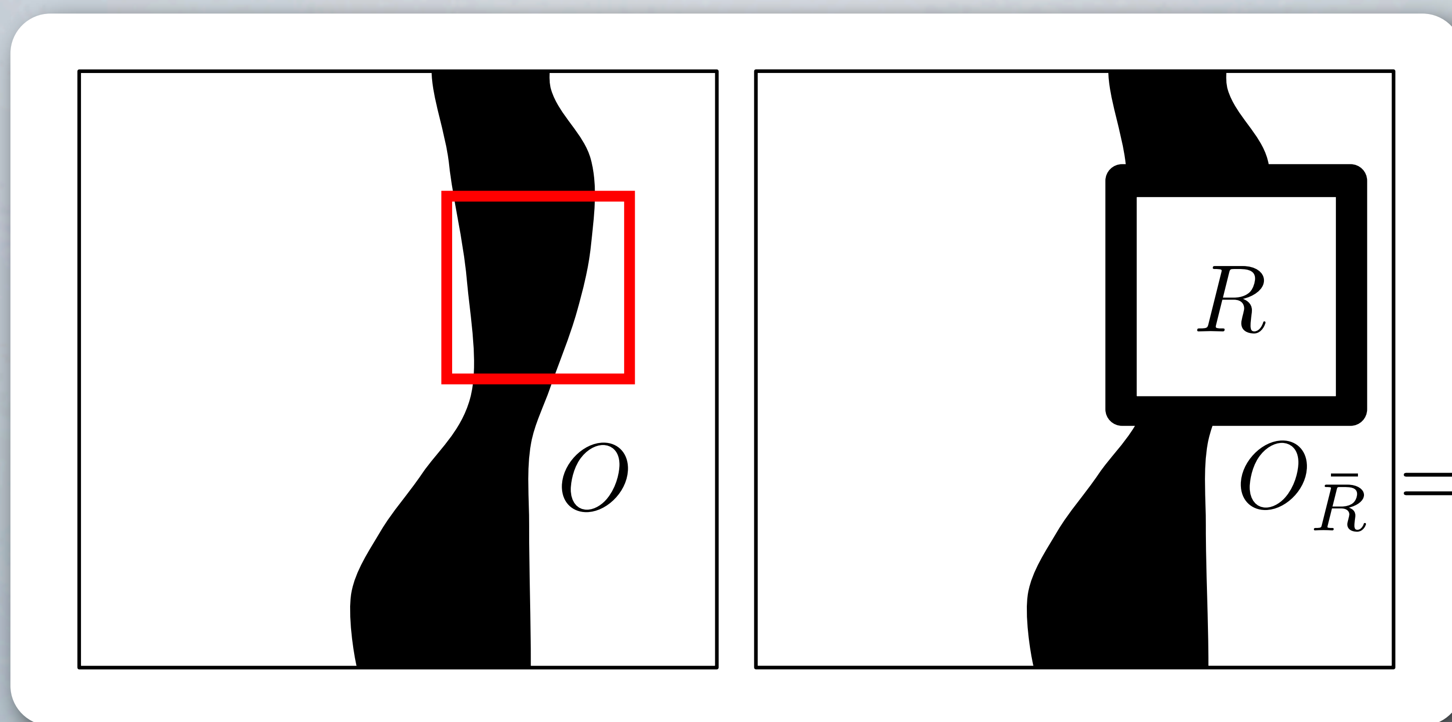
$$\mathcal{S} = \langle g_1, g_2, \ldots, g_m \rangle$$
$$[g_i, g_j] := g_i g_j g_i^\dagger g_j^\dagger = \mathbb{1}$$

- The code space: $\mathcal{C} = \{|\psi\rangle : P|\psi\rangle = |\psi\rangle \ \forall P \in \mathcal{S}\}$



Gottesman, D. (1997, May). Stabilizer Codes and Quantum Error Correction. quant-ph/9705052. Thesis @ Caltech.

CLEANING LEMMA



$$O_{\bar{R}} = \Lambda_R^{\dagger}(O)$$

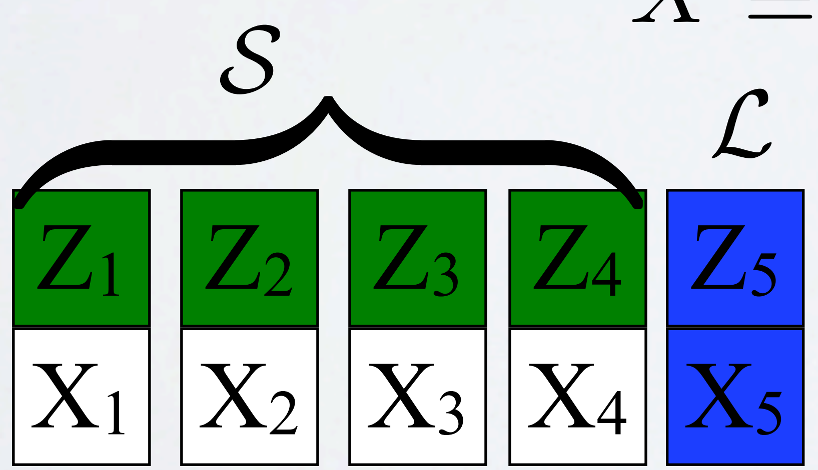
- For stabilizer codes: $O \in \mathcal{P} \Rightarrow O_{\bar{R}} \in \mathcal{P}$
- Bounded support growth for locally defined stabilizer codes.
- Union lemma: If two correctable regions don't share stabilizers their union is correctable.

Bravyi, S., & Terhal, B. (2009). A no-go theorem for a two-dimensional self-correcting quantum memory based on stabilizer codes. NJP, 11(4), 043029.

EXAMPLE: 5 QUBIT CODE

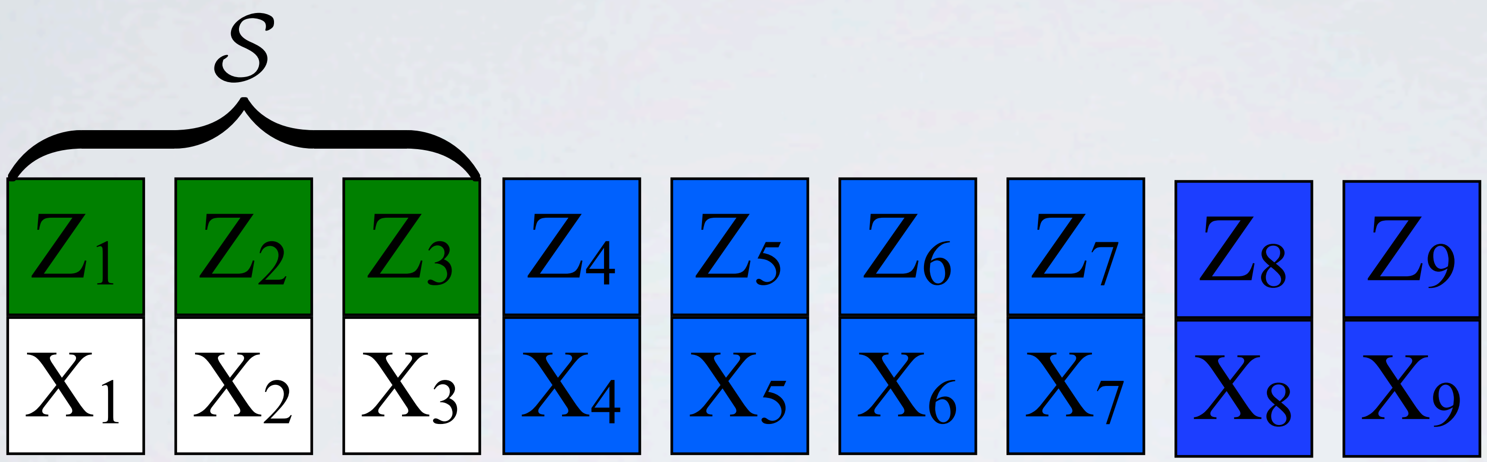
- Stabilizer group $\mathcal{S} = \langle ZIZXX, XZIZX, XXZIZ, ZXXZI \rangle$
- Detects up to two errors anywhere
- Encodes 1 logical qubit $\bar{X} = XXXXX \quad \bar{Z} = ZZZZZ$
- Suppose we loose second and fourth qubits

$$\bar{X} \equiv ZIXIZ \qquad \bar{Z} \equiv YIZIY$$



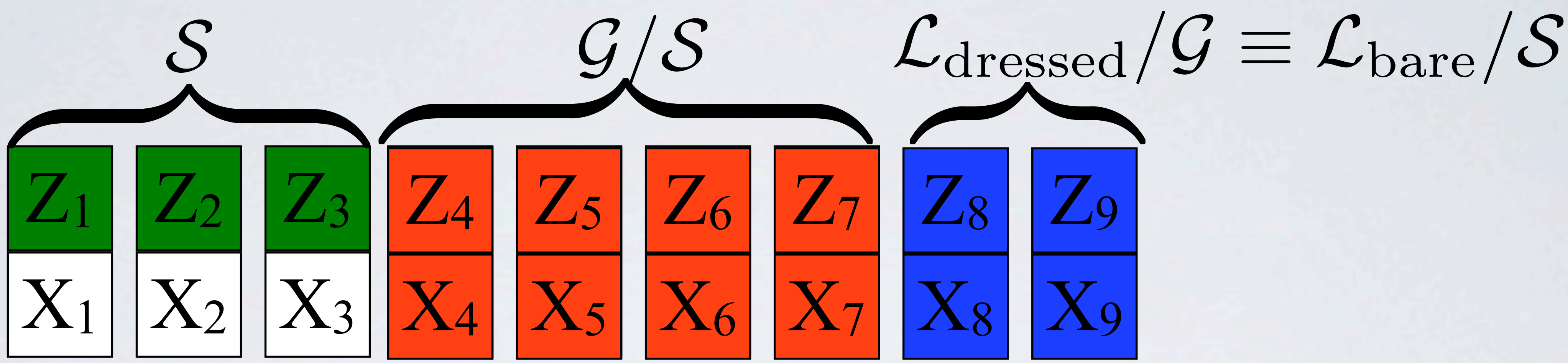
Laflamme, R., Miquel, C., Paz, J. P., & Zurek, W. H. (1996). Perfect Quantum Error Correcting Code. Physical Review Letters, 77(1), 198.

SUBSYSTEM CODES



Poulin, D. (2005). Stabilizer Formalism for Operator Quantum Error Correction. Physical Review Letters, 95(23), 230504–4.

SUBSYSTEM CODES



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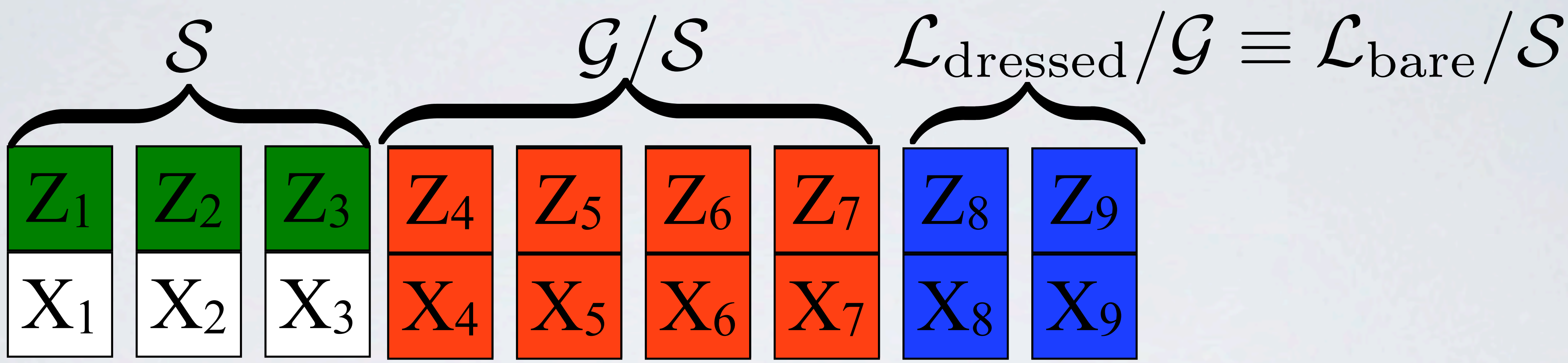
SUBSYSTEM CODES



- A gauge subgroup $\mathcal{G} = \langle g_1, g_2, \ldots, g_m \rangle \subseteq \mathcal{P}$ (Not necessarily commuting)

Hamiltonian $H \in K(\mathcal{G})$

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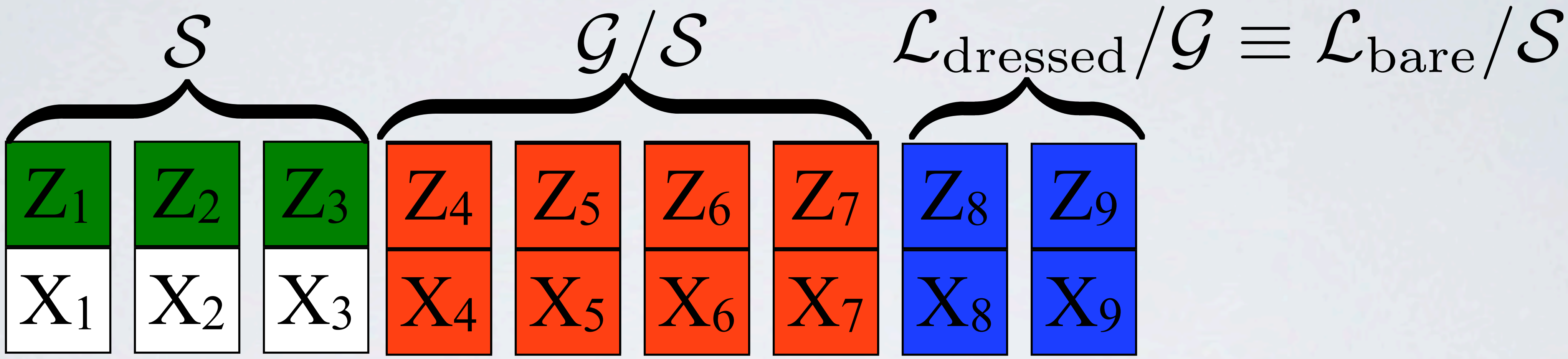
SUBSYSTEM CODES



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  (Not necessarily commuting)

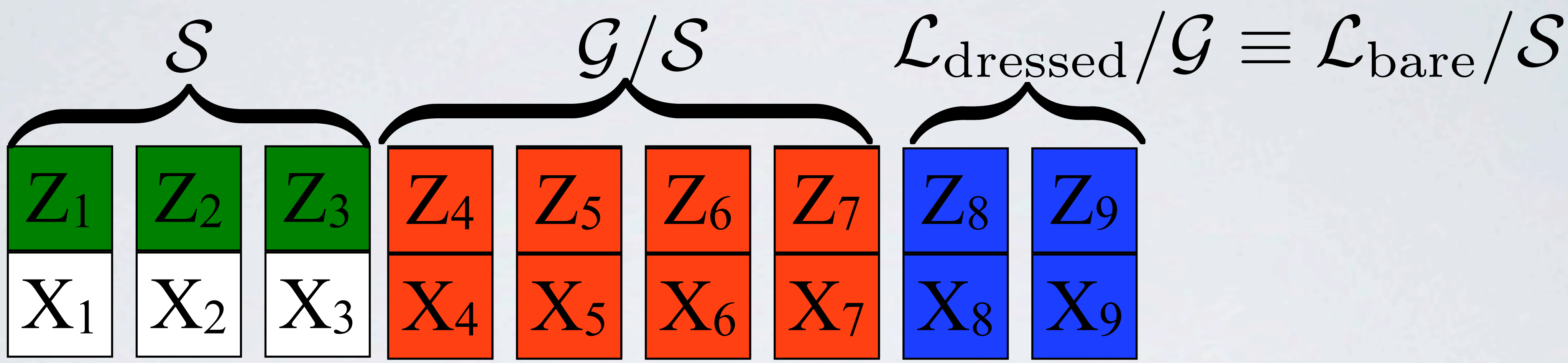
Hamiltonian
$H \in K(\mathcal{G})$

- Stabilizer subgroup, center of $\mathcal{G}$ (sign freedom):

$$\mathcal{S} = \{p \in \mathcal{G} : \forall g \in \mathcal{G},\ [p, g] = 0\}$$

Poulin, D. (2005). Stabilizer Formalism for Operator Quantum Error Correction. Physical Review Letters, 95(23), 230504–4.

SUBSYSTEM CODES



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$$\mathcal{S} = \{p \in \mathcal{G} : \forall g \in \mathcal{G},\ [p, g] = 0\}$$

- Bare logical operators:

$$\mathcal{L}_{\text{bare}} = \{p \in \mathcal{P} : \forall g \in \mathcal{G},\ [p, g] = 0\}$$

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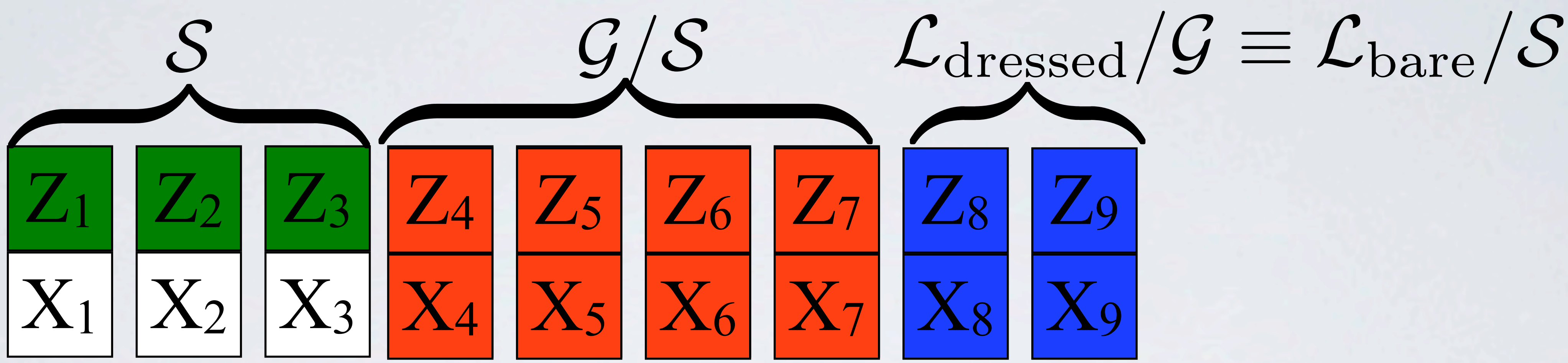
SUBSYSTEM CODES



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Hamiltonian
$H \in K(\mathcal{G})$

- Stabilizer subgroup, center of $\mathcal{G}$ (sign freedom):

$$\mathcal{S} = \{p \in \mathcal{G} : \forall g \in \mathcal{G},\ [p, g] = 0\}$$

- Bare logical operators:

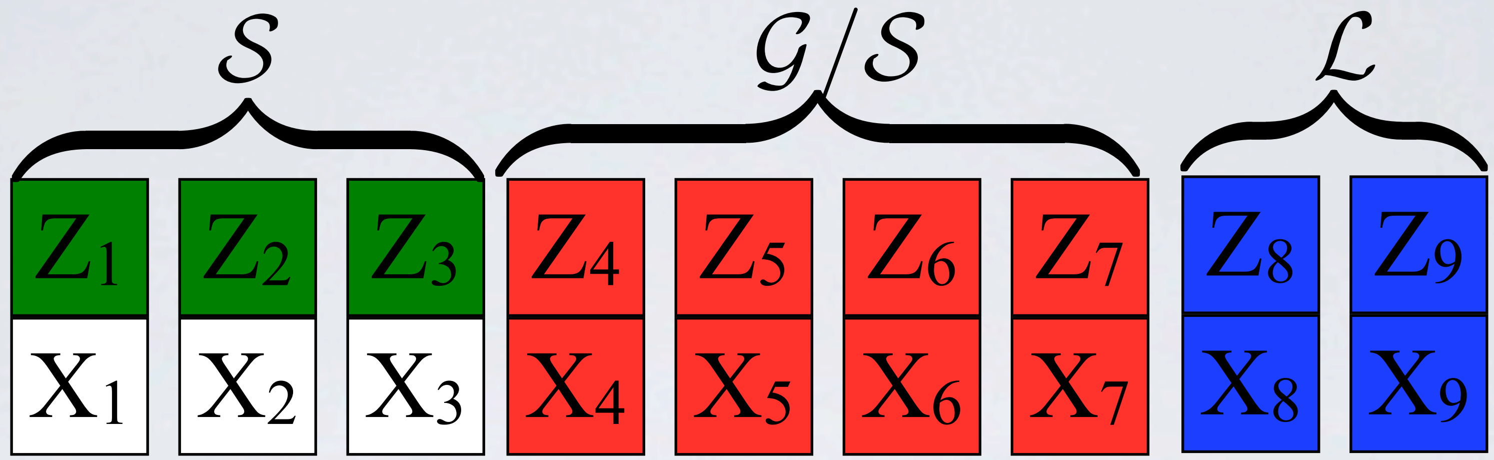
$$\mathcal{L}_{\text{bare}} = \{p \in \mathcal{P} : \forall g \in \mathcal{G},\ [p, g] = 0\}$$

- Dressed logical operators:

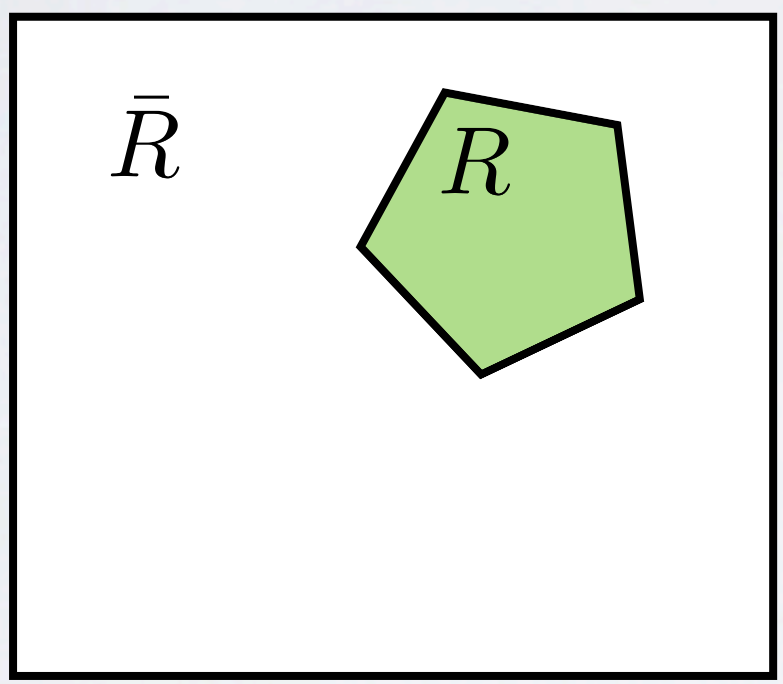
$$\mathcal{L}_{\text{dressed}} = \{p \in \mathcal{P} : \forall g \in \mathcal{S},\ [p, g] = 0\}$$

Poulin, D. (2005). Stabilizer Formalism for Operator Quantum Error Correction. Physical Review Letters, 95(23), 230504–4.

SUBSYSTEM CODES DUALITY



$$n_{\text{dressed}}(R) + n_{\text{bare}}(\bar{R}) = 2\text{dim}(\mathcal{L})$$



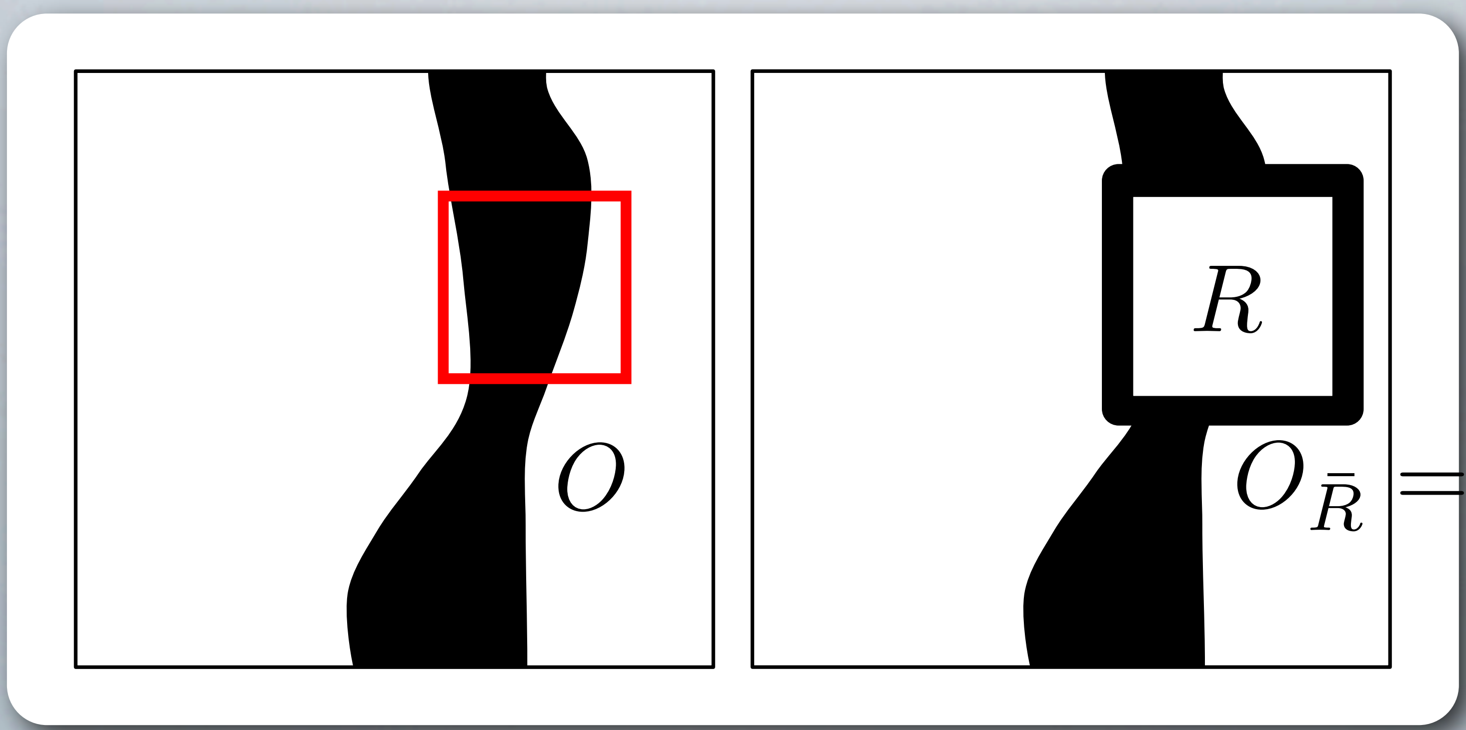
$$n_{\text{bare}}(R) \leq n_{\text{dressed}}(R)$$

$$\text{Correctable}(R) \iff n_{\text{dressed}}(R) = 0$$

$$\text{Dresscleanable}(R) \iff n_{\text{bare}}(R) = 0$$

Bravyi, S. (2011). Subsystem codes with spatially local generators. Physical Review A, 83(1), 012320.

SUBSYSTEM CLEANING



$O_{\bar{R}} = \Lambda_R^\dagger(O)$

- Also for subsystem codes codes: $O \in \mathcal{P} \Rightarrow O_{\bar{R}} \in \mathcal{P}$
- Bounded support growth of dressed operators for locally generated gauge group.
- Union lemmas for bare and dressed cleanable regions.
  Warning: local gauge operators may yield non-local stabilizers

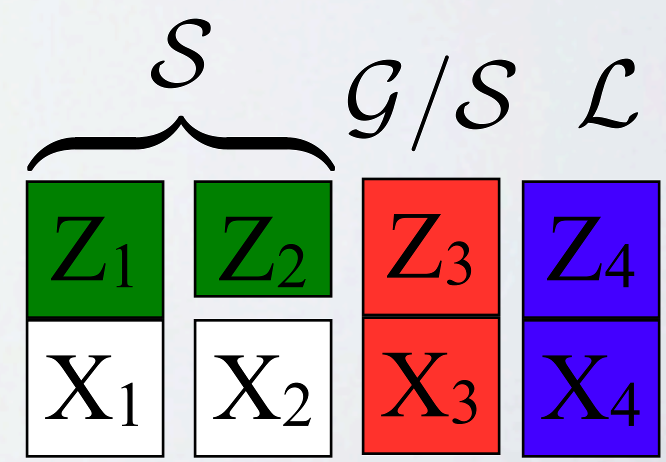
Bravyi, S., & Terhal, B. (2009). A no-go theorem for a two-dimensional self-correcting quantum memory based on stabilizer codes. NJP, 11(4), 043029.

EXAMPLE: 4 QUBIT CODE

- Gauge group: $\mathcal{G} = \langle XXII, IIXX, IZZI, ZIIZ \rangle$
- Stabilizer group $\mathcal{S} = \langle XXXX, ZZZZ \rangle$
- Detects one error anywhere. Corrects none.
- Encodes 1 logical qubit $\bar{X} = XIIX \qquad \bar{Z} = ZZII$
- Suppose we loose the first qubit (correctable)

$$\bar{X} \equiv IXXI \qquad \bar{Z} \equiv IIZZ$$

- Dress-clean (1st and 3rd qubits)



$$\bar{X}_{dressed} \equiv IXIX \qquad \bar{Z}_{dressed} \equiv IZIZ$$

Bacon, D., Brown, K. R., & Whaley, K. B. (2001). Coherence-Preserving Quantum Bits. Physical Review Letters, 87(24), 247902.

REDERIVING BRAVYI-KÖNIG

CLASSIFICATION OF TOPOLOGICALLY

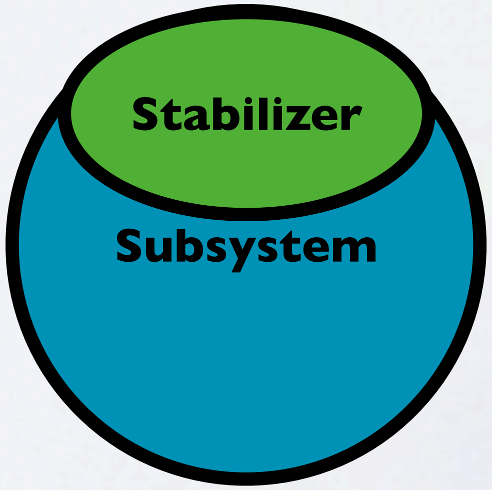
subsystem

PROTECTED GATES ON ~~STABILIZER~~ CODES

and important observations

Bravyi, S., & König, R. (2013).
Classification of Topologically Protected Gates for Local Stabilizer Codes.
Physical Review Letters, 110(17), 170503.

Pastawski, F., & Yoshida, B. (2014).
Fault-tolerant logical gates in quantum error-correcting codes.
arXiv:1408.1720

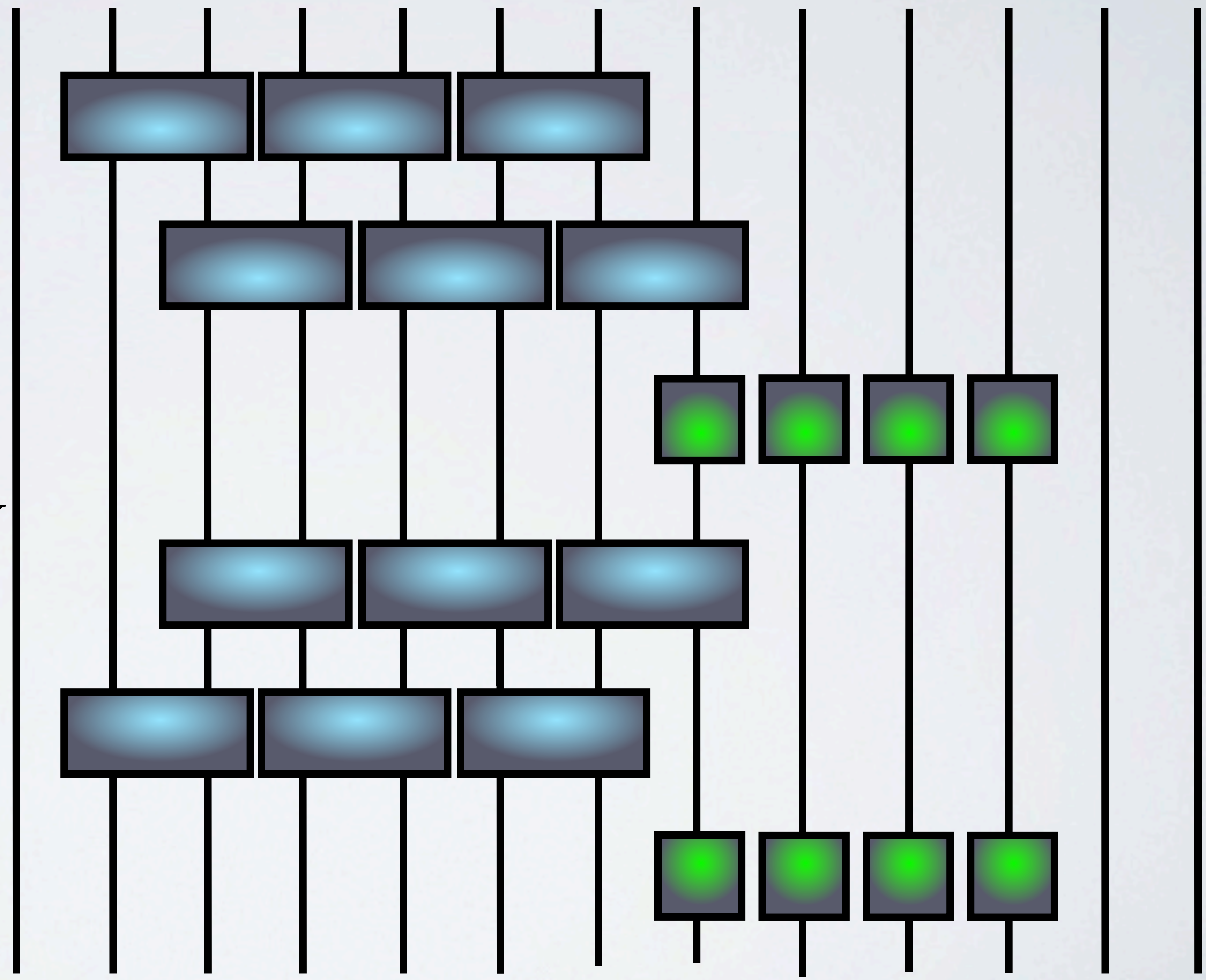


BK FOR SUBSYSTEM CODES

Theorem: Every transverse dressed logical operator U supported on the union of a **correctable** region Λ_0 and n **dressed-cleanable** regions $\{\Lambda_j\}_{(j\in[1,n])}$, must correspond to a logical operator in $\mathcal{P}_n$.

COMMUTATOR CLEANING

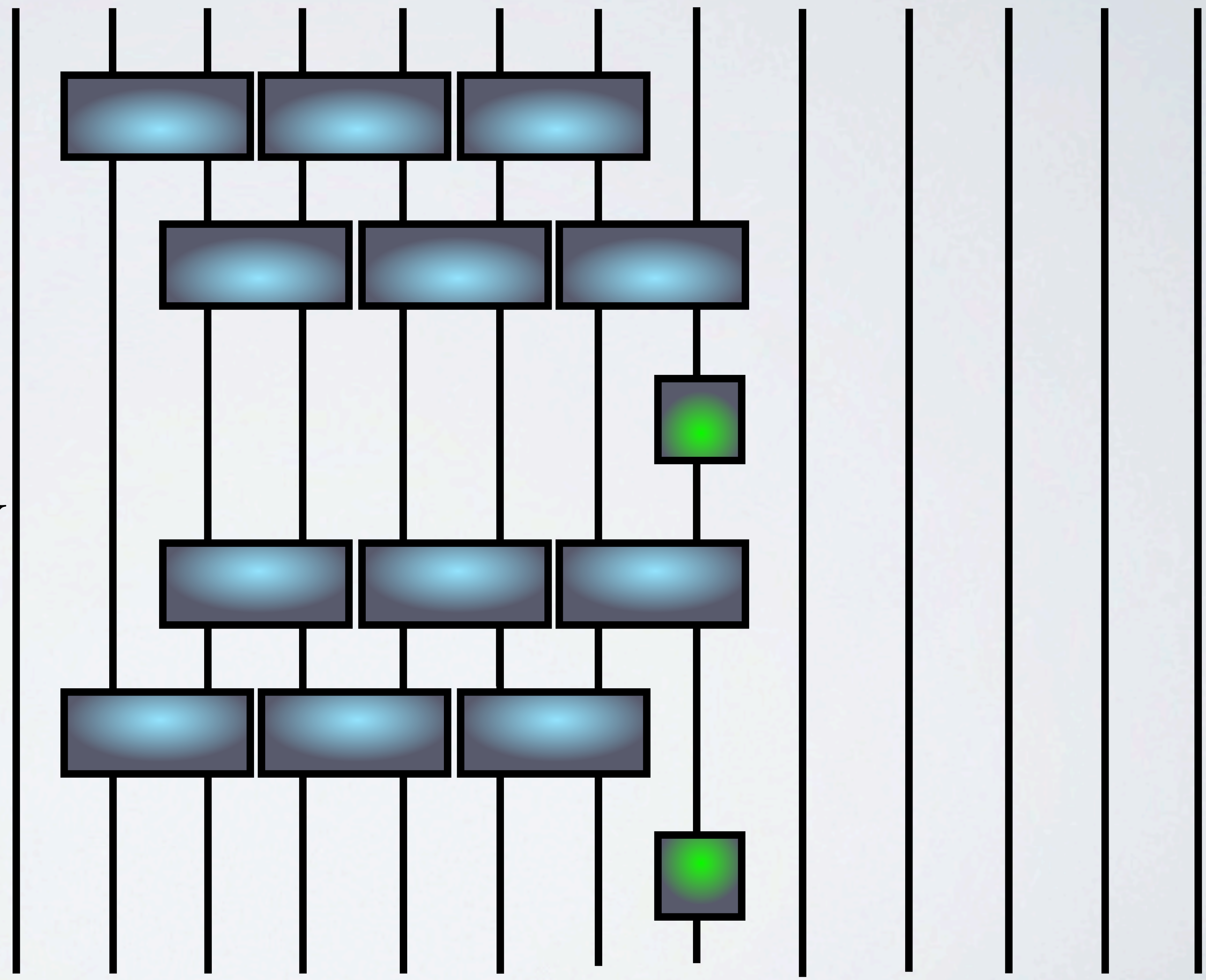
$$[U,V] = UVU^\dagger V^\dagger =$$



$$R_{[U,V]} \subseteq R_U \cup R_V$$

COMMUTATOR CLEANING

$$[U,V] = UVU^\dagger V^\dagger =$$



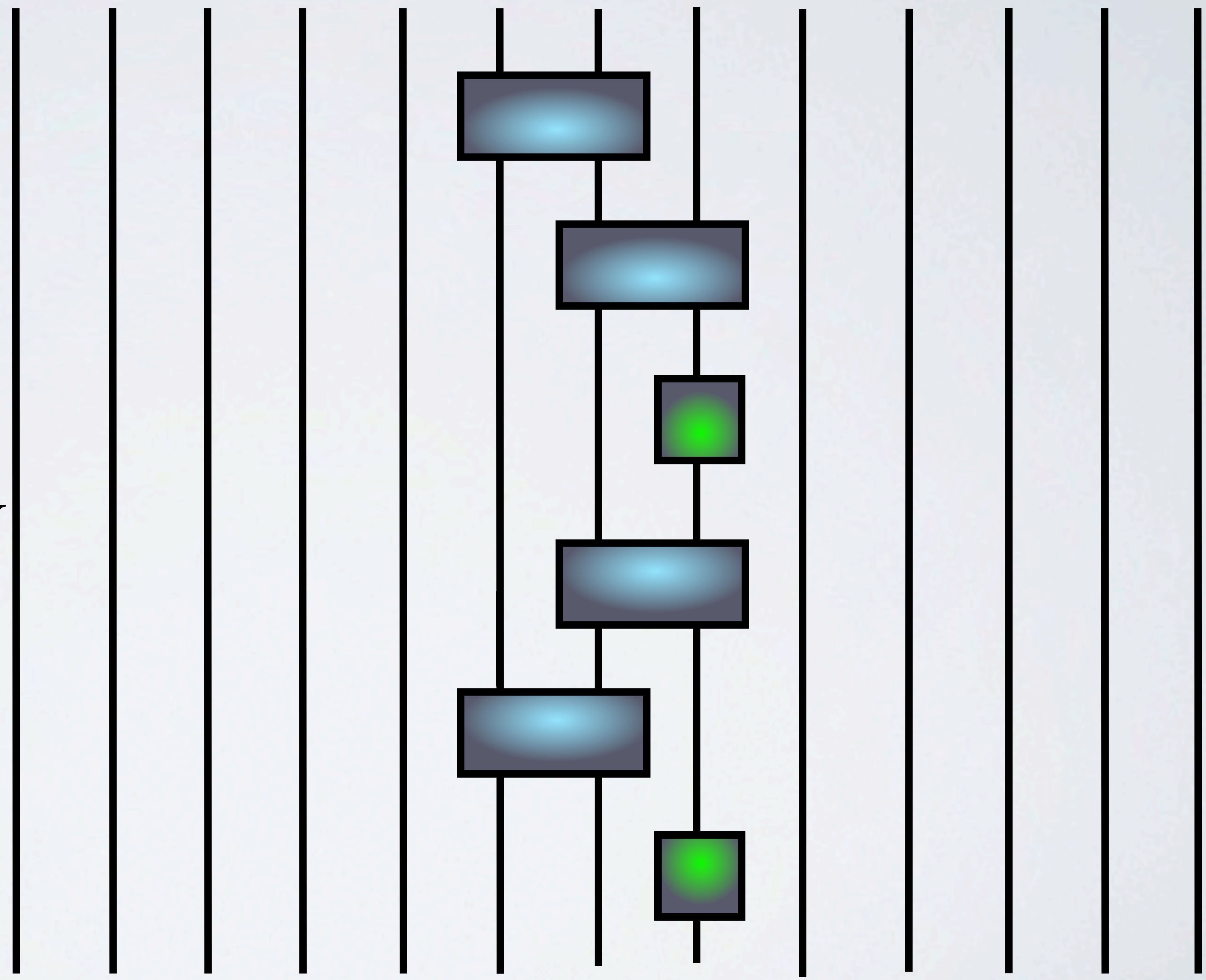
$$R_{[U,V]} \subseteq R_U \cup R_V$$

V is transverse.

$$R_{[U,V]} \subseteq R_U$$

COMMUTATOR CLEANING

$[U, V] = UVU^\dagger V^\dagger =$



$R_{[U,V]} \subseteq R_U \cup R_V$

V is transverse.

$R_{[U,V]} \subseteq R_U$

U also transverse.

$R_{[U,V]} \subseteq R_U \cap R_V$

- Consider arbitrary Pauli logical operators $V_0, V_1, \ldots V_m$.

	R_0,	R_1,	R_2,	...	R_{m-1},	R_m	Hierarchy
V_1	✓		✓	...	✓	✓	Pauli
⋮				⋮			
V_m	✓	✓	✓	...	✓		

group commutator : $[U,V]=UVU^{-1}V^{-1}$

- Consider arbitrary Pauli logical operators $V_0, V_1, \ldots V_m$.

	R_0,	R_1,	R_2,	...	R_{m-1},	R_m	Hierarchy
V_1	✓		✓	...	✓	✓	Pauli
$\vdots$				$\vdots$			
V_m	✓	✓	✓	...	✓		
U_m	✓	✓	✓	...	✓	✓	
$U_{m-1}=[U_m,V_m]$	✓	✓	✓	...	✓		

group commutator : $[U,V]=UVU^{-1}V^{-1}$

- Consider arbitrary Pauli logical operators $V_0, V_1, \dots V_m$.

	R_0,	R_1,	R_2,	...	R_{m-1},	R_m	Hierarchy
V_1	✓		✓	...	✓	✓	Pauli
⋮				⋮			
V_m	✓	✓	✓	...	✓		
U_m	✓	✓	✓	...	✓	✓	
$U_{m-1}=[U_m,V_m]$	✓	✓	✓	...	✓		
⋮				⋮			
$U_2=[U_3,V_3]$	✓	✓	✓	...			
$U_1=[U_2,V_2]$	✓	✓		...			
$U_0=[U_1,V_1]$	✓			...			

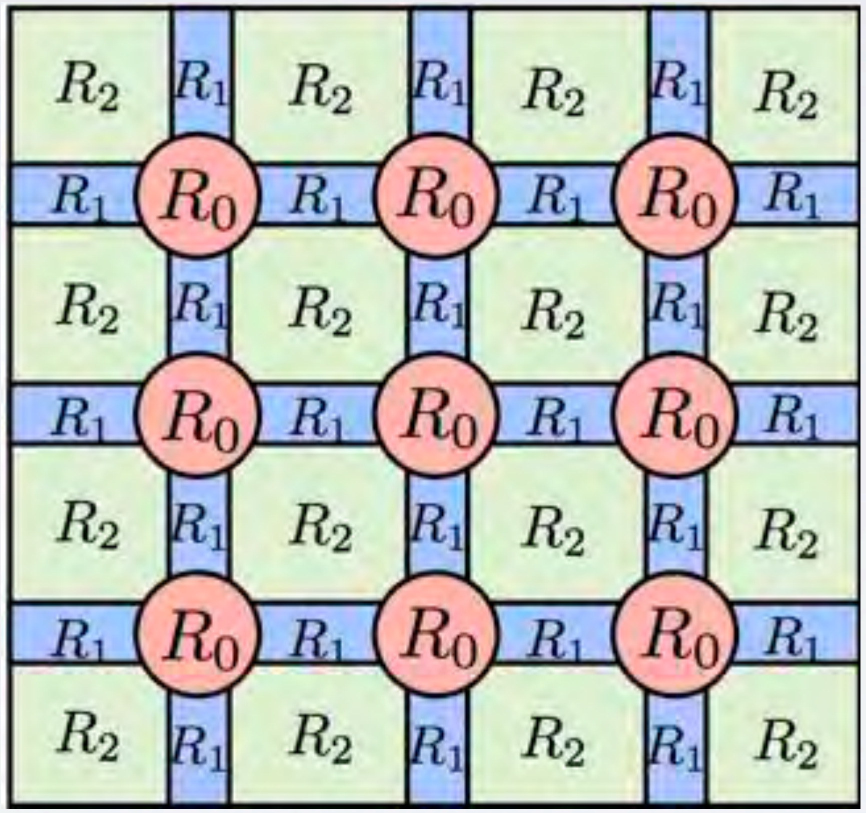
group commutator : $[U,V]=UVU^{-1}V^{-1}$

- Consider arbitrary Pauli logical operators V_0, V_1, ... V_m.

	R_0,	R_1,	R_2,	...	R_{m-1},	R_m	Hierarchy
V_1	✓		✓	...	✓	✓	Pauli
⋮				⋮			
V_m	✓	✓	✓	...	✓		
U_m	✓	✓	✓	...	✓	✓	$\boxed{P_m}$ ← goal
$U_{m-1}=[U_m,V_m]$	✓	✓	✓	...	✓		P_{m-1}
⋮				⋮			
$U_2=[U_3,V_3]$	✓	✓	✓	...			P_2 (Clifford gr.)
$U_1=[U_2,V_2]$	✓	✓		...			P_1 (Pauli)
$U_0=[U_1,V_1]$	✓			...			Complex phase

group commutator : $[U,V]=UVU^{-1}V^{-1}$

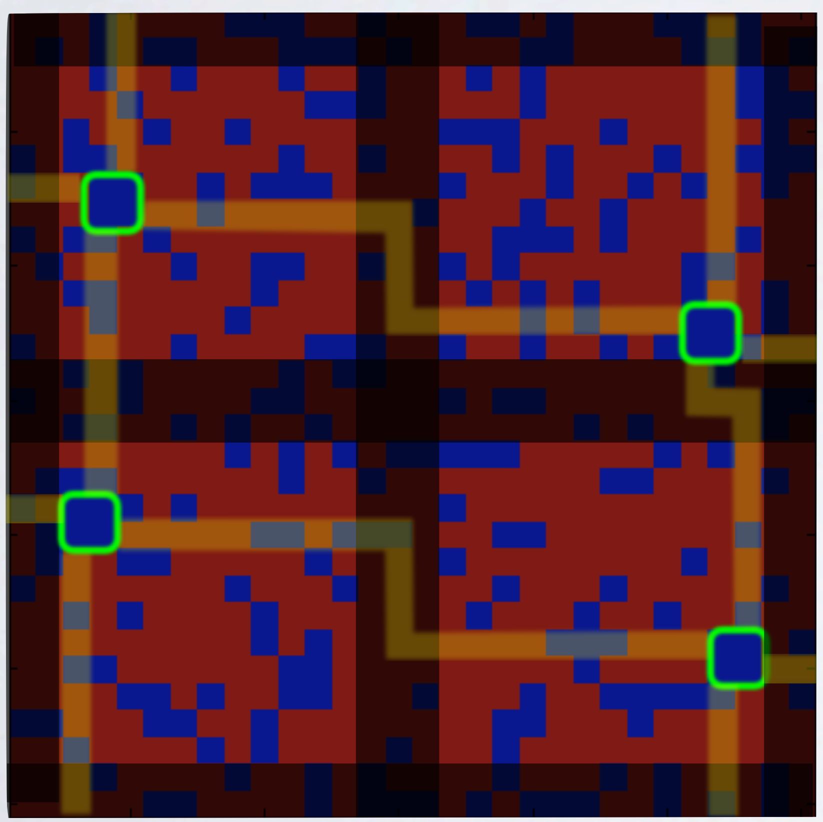
BK REGION DECOMPOSITION

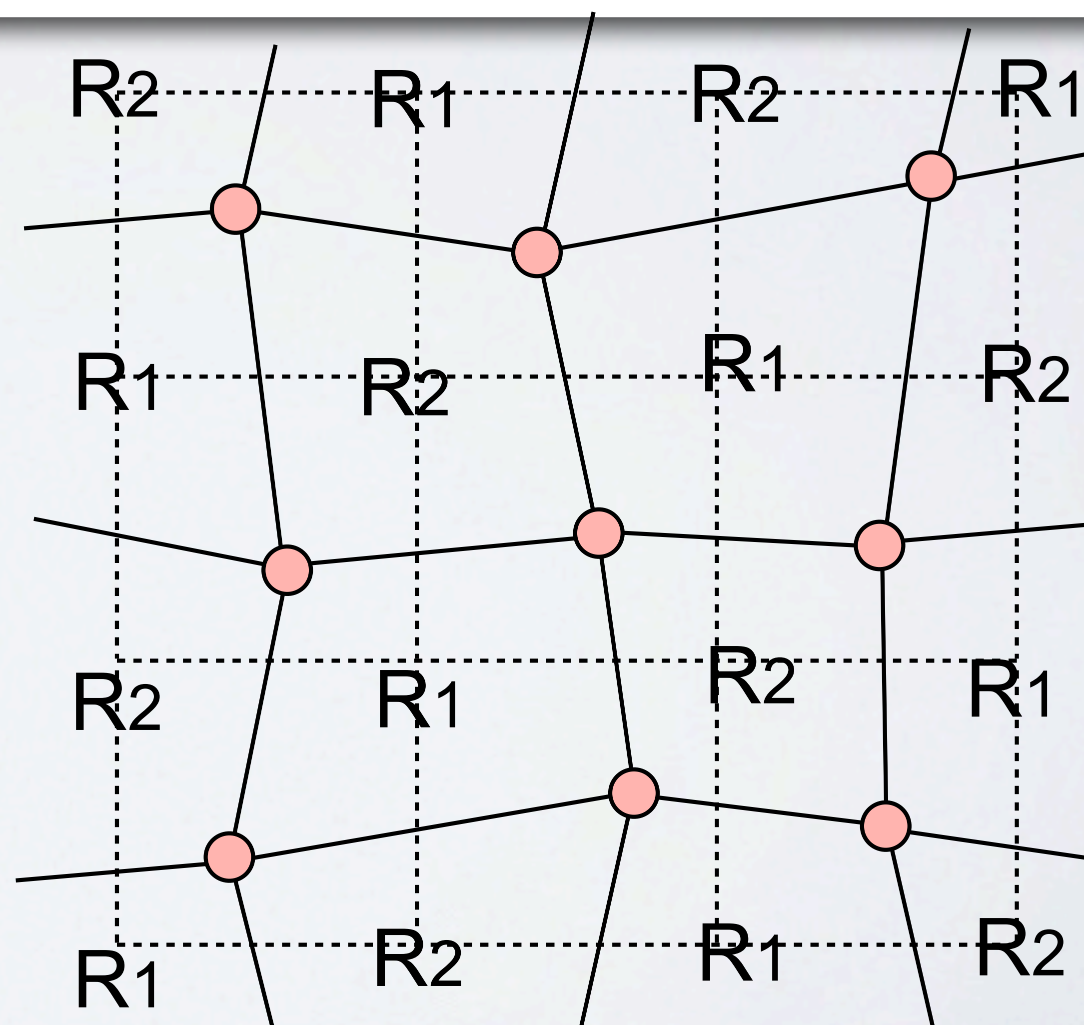


Qubits participating in a D dimensional stabilizer code may be partitioned into D+1 correctable regions.

GEOMETRIC OBSERVATION

Observation: Every D dimensional region in a locally generated subsystem code with threshold and log growing distance may be partitioned into a correctable region Λ_0 and D dressed-cleanable regions $\{\Lambda_j\}_{(j\in[1,D])}$.

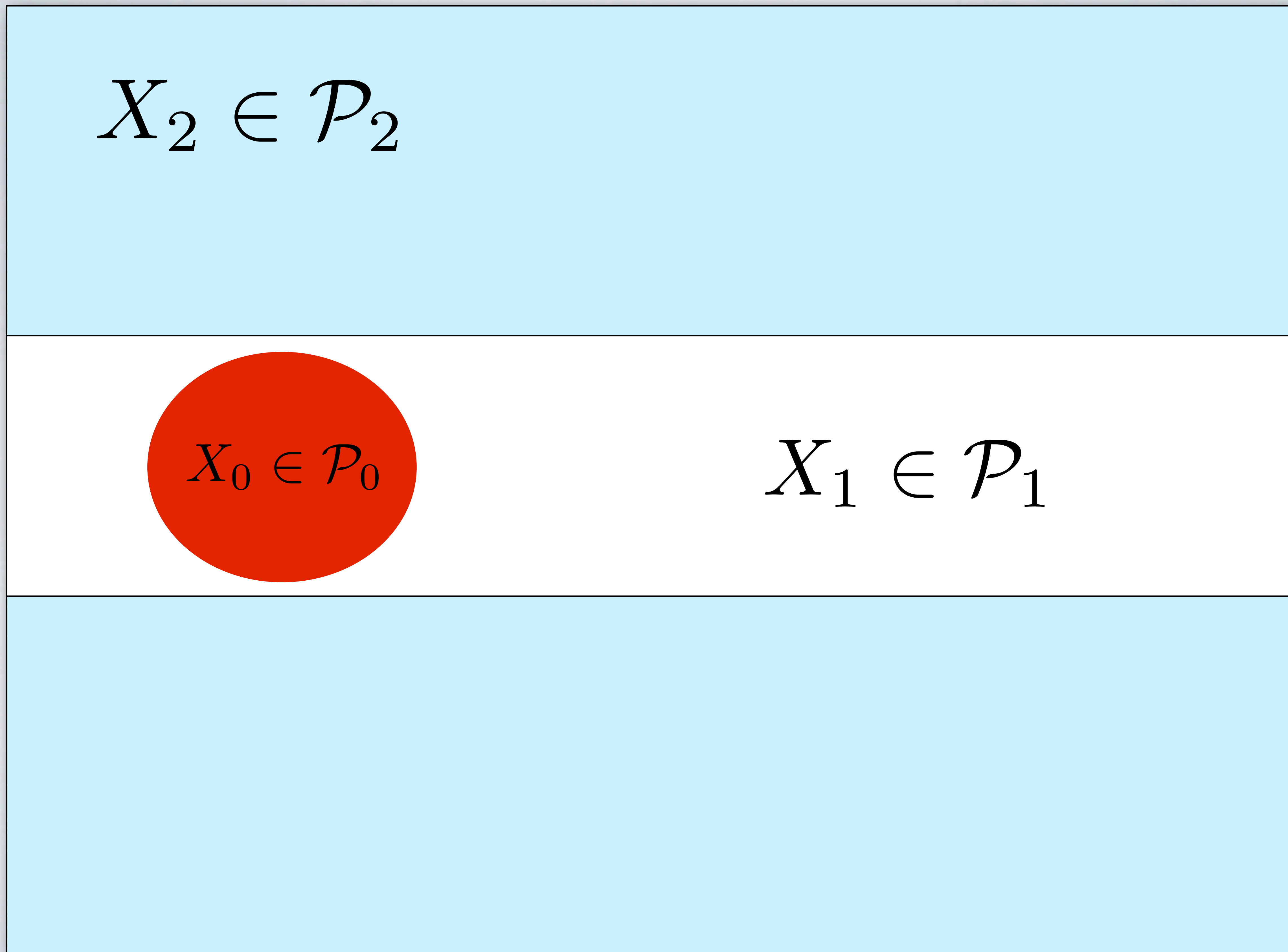




GEOMETRY CONSTRAINED LOGICAL OPERATORS

Corollary: Every transverse dressed logical operator U supported on a D dimensional region of a locally defined subsystem code with an erasure threshold and logarithmic diverging distance must be in $\mathcal{P}_D$.

Also extends to U with constant depth circuit implementations. (like Bravyi-König)


$X_2 \in \mathcal{P}_2$
$X_0 \in \mathcal{P}_0$
$X_1 \in \mathcal{P}_1$

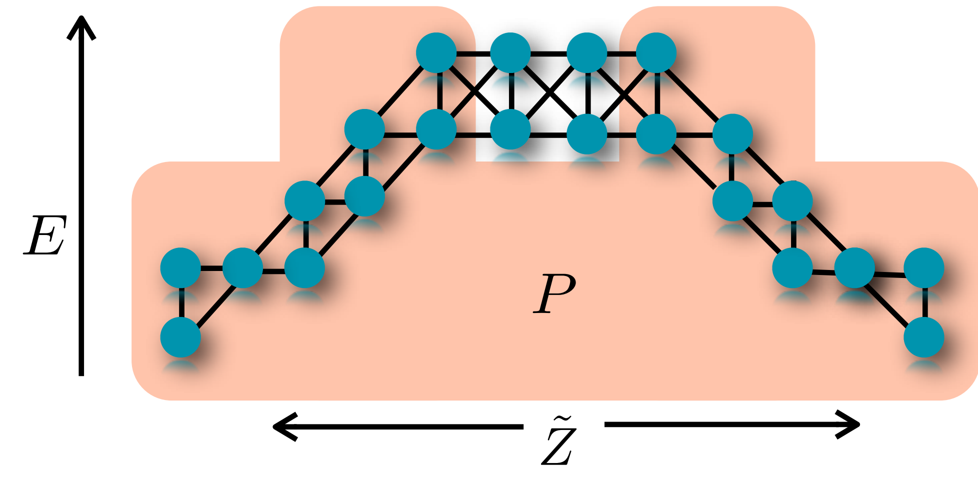
OBSERVATIONS

TRADEOFF WITH SELF-CORRECTION

SELF-CORRECTION & THE NO-STRINGS RULE

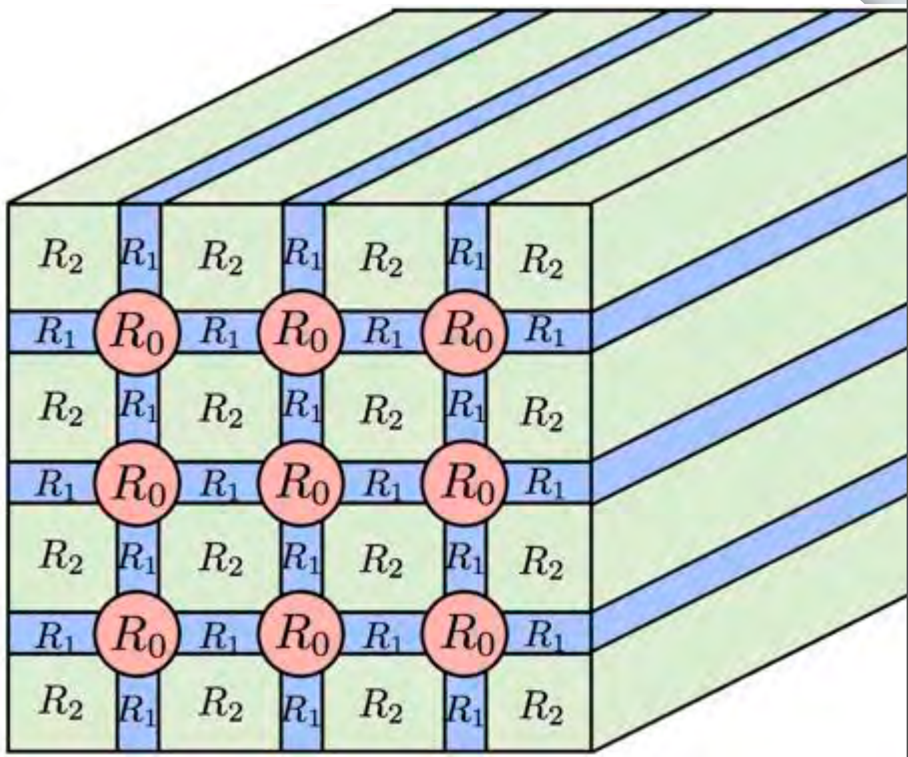
Folklore:

- For thermally stable (self-correcting) memory a growing energy barrier is expected to be necessary.
- Logical operators supported on a string may be implemented sequentially excluding such a barrier.
- Stringlike regions should be correctable



Haah, J. (2011). Local stabilizer codes in three dimensions without string logical operators. http://arxiv.org/abs/1101.1962
Landon-Cardinal, O., & Poulin, D. (2013). Local Topological Order Inhibits Thermal Stability in 2D. Physical Review Letters, 110(9), 090502.

NO-STRINGS RULE & DIMENSION REDUCTION



Observation: Every D dimensional region in a subsystem code with
- **local stabilizer generators**
- growing distance
- **no-string rule**

may be partitioned into a correctable region Λ_0 and D-1 dressed-cleanable regions $\{\Lambda_j\}_{(j\in[1,D\text{-}1])}$.

COROLLARY

- **Haah code, Michnicki code, Kim code, Brell code** and all other no string codes in 3D have no non-clifford logical operators.

CODE DISTANCE TRADEOFF

- $d > L^n$**:** A regular lattice and large distance implies a generalized no-string (no slab) rule.

- We get an upper bound for code distance from the converse

$$[U] \in \mathcal{P}_n \Rightarrow d \leq O(L^{D+1-n})$$

TRADEOFF WITH ERASURE THRESHOLD

ERASURE THRESHOLD

- **Erasure threshold p_e:** An i.i.d. random subset of qubits taken with probability $p < p_e$ is correctable with high probability.
- There is a partition into n correctable regions $n := \left\lceil \frac{1}{p_e} \right\rceil$
- Transverse logicals are in $\mathcal{P}_{n-1}$
- Identify trade-off of transverse gates with erasure threshold p_e

$$[U] \in \mathcal{P}_n \Rightarrow p_e \leq 1/n$$

- n-th level Cliffords require linear weight stabilizers in n (Pryadko)

Note that:

loss threshold ≥ error threshold

Pryadko Leonid (Personal communication)

Recover result for fault-tolerant subsystem codes with local gauge group in D-dimensions.

Identify trade-off with code distance d

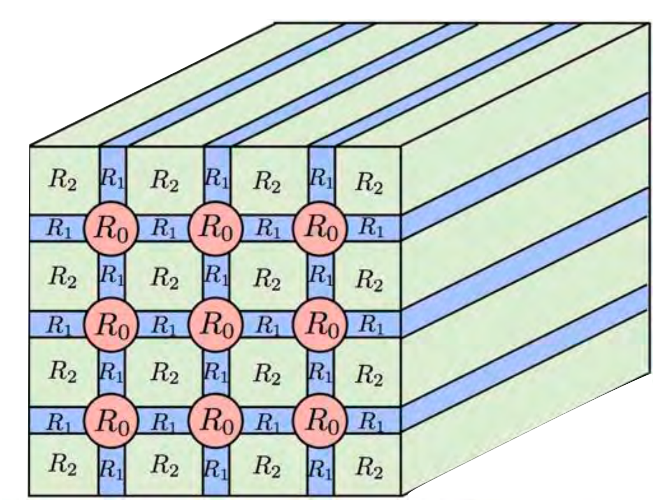
$$[U] \in \mathcal{P}_n \Rightarrow d \leq O(L^{D+1-n})$$

Summary: Observations and extensions of BK results to subsystem codes. Requires threshold & d > log

Identify trade-off of transverse gates with erasure threshold p_{err}

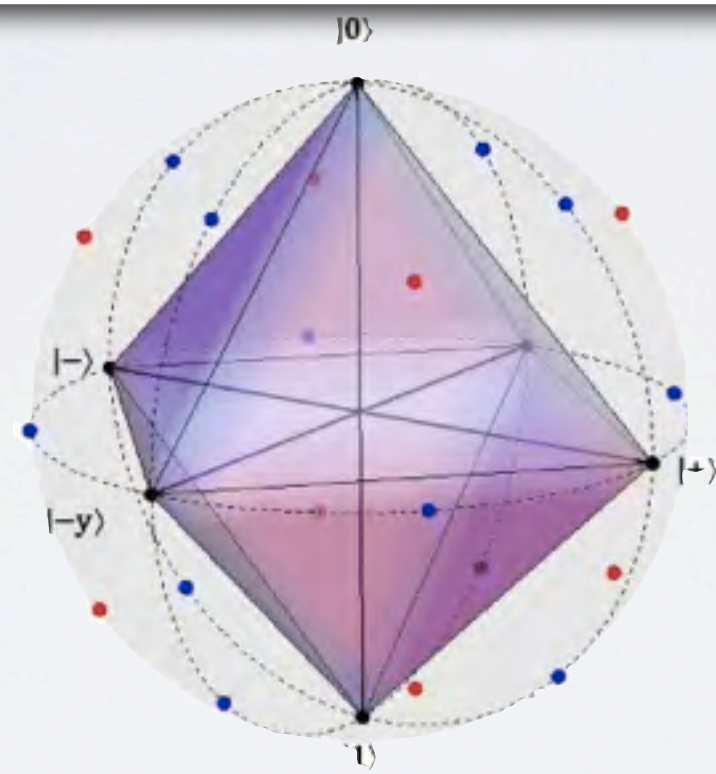
$$[U] \in \mathcal{P}_n \Rightarrow p_{err} \leq 1/n$$

Strengthen result when imposing energy barrier through a no-string rule



HAMILTONIAN PHASES VS. STATE PHASES

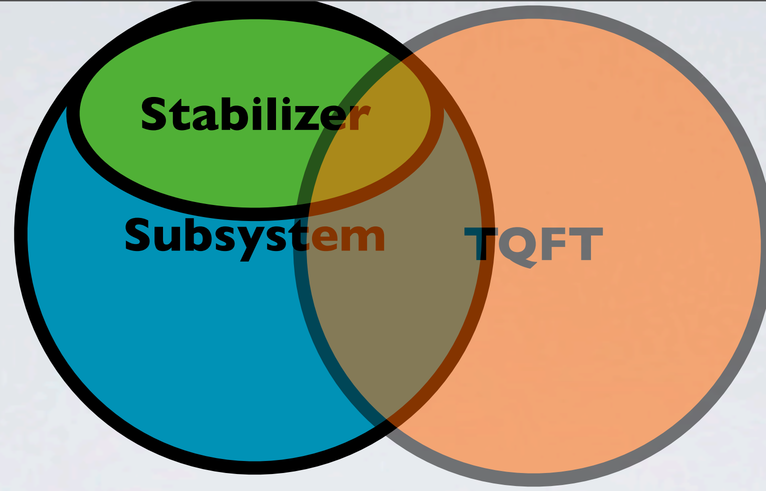
Observation: In 2D stabilizer codes, encoded magic and stabilizer states are in different phases.



Observation: Translation invariant Hamiltonians can adiabatically, prepare stabilizer code states efficiently.

CONCLUSIONS

- Local processing is not enough for universality.
- Require non-local quantum (or classical)
    - Measurement and feedback dependent on non-local classical processing
- Outlook: Topological quantum field theories :)
  LDPC codes.
  Non-local-gates.
  Classify the subgroups of $\mathcal{P}_3$ (or even $\mathcal{P}_n$).
  Interplay with fault tolerance techniques



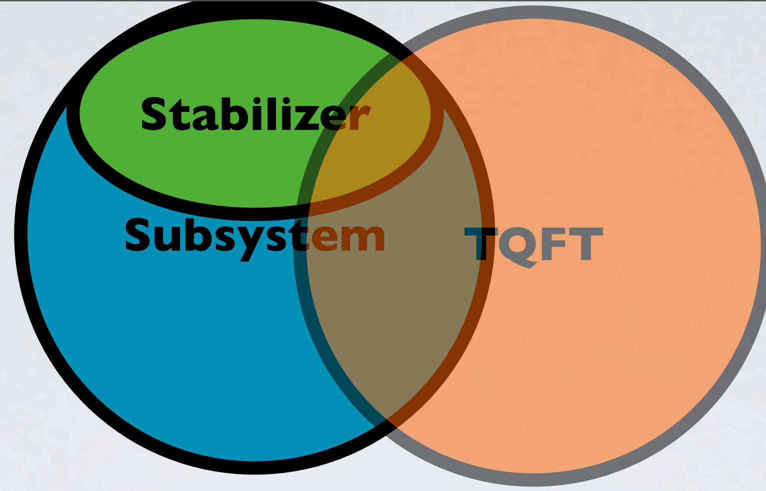
REDERIVING BRAVYI-KÖNIG

CLASSIFICATION OF LOCAL GATES ON

Topological quantum field theories ~~STABILIZER CODES~~

TQFT codes
Topological quantum field theories (vacua)

Beverland, M. E., König, R., Pastawski, F., Preskill, J., & Sijher, S. (2014). Protected gates for topological quantum field theories. arXiv:1409.3898



REDERIVING BRAVYI-KÖNIG

CLASSIFICATION OF LOCAL GATES ON Topological quantum field theories ~~STABILIZER CODES~~

TQFT codes Topological quantum field theories (vacua)

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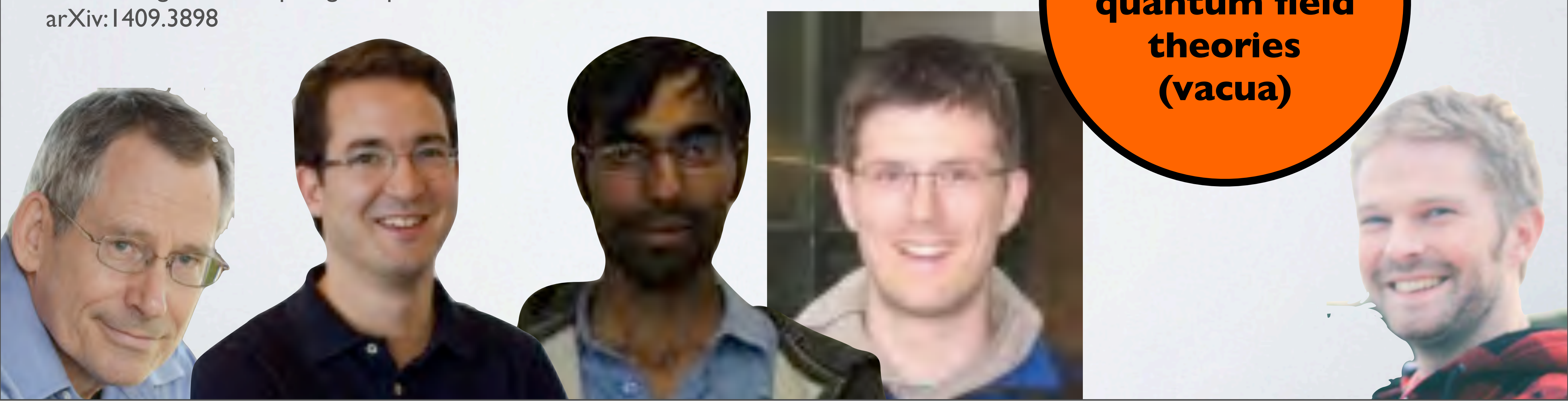

THANK YOU!