New existence bounds for decoding transition with Q-LDPC codes: percolation on hypergraphs

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QEC14: Dec 16, 2014

\[ p_e + (1 - p_e)[4p_x (1 - p_x)]^{1/2} < (w_z - 1)^{-1} \]

\[ 2[4q(1 - q)]^{1/2} + w_z \left\{ p_e + (1 - p_e)[4p_x (1 - p_x)]^{1/2} \right\} < 1 \]

Ilya Dumer (UCR)
Alexey Kovalev (UNL)
Kathleen Hamilton (UCR)
arXiv:1208.2317
arXiv:1311.7688
arXiv:1405.0050
\(^1\)arXiv:1405.0348 & new work
New existence bounds for decoding transition with Q-LDPC codes: percolation on hypergraphs

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• Introduction: SAW-based bound for the surface codes
• Old bound for Q-LDPC codes with log distance
• New bounds: count irreducible undetectable operators
• Conclusions and open problems

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Decoding threshold $p_c$: Consider an infinite family of error correcting codes. With probability $p$ for independent errors per (qu)bit, at $p < p_c$, a large enough code can correct all errors with success probability $P \to 1$, but not at $p > p_c$.

Example: code family with finite relative distance $\delta = d/n$. A code can detect any error involving $w < d$ (qu)bits, and distinguish between any two errors involving $w < d/2$ qubits each. For such a family, $p_c \geq \delta/2$.

In practice, this does not quite work since such codes have stabilizer generators of weight $\sim n$: measuring syndrome is hard.

All known code families with finite-weight stabilizer generators have distance scaling logarithmically or as a sublinear power of $n$.

Zero-rate codes: toric (Kitaev) color (Bombin et al.)

Surface codes

Family of codes invented by Alexey Kitaev (orig: toric codes)

Stabilizer generators: plaquette $A_\square = ZZZZ$ and vertex $B_+ = XXXX$ operators (this is a CSS code).

toric code $[[98, 2, 7]]$
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\[
\text{toric code } [[98, 2, 7]]
\]
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\begin{array}{cccc}
Z & Z & Z & Z \\
\end{array}$ and vertex $B_+ = 
\begin{array}{cccc}
X & X & X & X \\
\end{array}$ operators (this is a CSS code).

Detectable errors: have open $X$ chains along dual lattice or open $Z$ chains on the original lattice

Undetectable error: only closed chains

**Trivial** undetectable error: topologically trivial loops

**Bad** undetectable error: topologically non-trivial loop $\Rightarrow$ Code distance $d = L \propto \sqrt{n}$.

$[[n = 2L^2, k = 2, d = L]]$
Surface codes: finite decoding threshold

Distance scales as $d \propto n^{1/2}$, meaning zero relative distance $\delta \propto n^{-1/2}$, $n \to \infty$. Is there a finite decoding threshold?

Yes! [Dennis, Kitaev, Landahl & Preskill, 2002]

- Counting topologically non-trivial chains
- Mapping to the Ising model with bond disorder

\[ \text{toric code } [[98, 2, 7]] \]
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Erasures: unrecoverable chain len. $\ell \geq d$:

$$Q_\ell \leq np^\ell \#(\text{SAW}_\ell) \leq n (3p)^\ell$$

Uncorrectable error: such a chain more than half-filled with errors. Probability:

$$P_\ell \leq n \#(\text{SAW}_\ell) \sum_{m \geq \lceil \ell/2 \rceil} \binom{\ell}{m} p^m (1 - p)^{\ell-m}$$

$$P_\ell \leq n 3^\ell \times 2^\ell [p(1 - p)]^{\ell/2}$$

Neither happens at sufficiently small $p$!
General \((h, w)\)-limited Q-LDPC codes

**Example:** hypergraph product code constructed from \([7, 3, 4]\) cyclic code. Column weights \(\leq h = 3\), row weights \(\leq w = 6\).

\[
G_x = \begin{pmatrix}
1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \ldots & 1 & 0 & 0 & 0 & \ldots \\
0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & \ldots & 0 & 1 & 0 & 0 & \ldots \\
0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & \ldots & 0 & 0 & 1 & 0 & \ldots \\
0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & \ldots & 0 & 0 & 0 & 1 & \ldots \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & \ldots & 0 & 0 & 0 & 0 & \ldots \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & \ldots & 0 & 0 & 0 & 0 & \ldots \\
\end{pmatrix}
\]

Observation: for small \(p\), errors can be separated into clusters which affect different subsets of generators.

Here, each qubit has up to \(z \equiv h(w - 1)\) neighbors.

Formation of large clusters can be viewed as percolation on a graph with vertex degrees bounded by \(z\).
Threshold theorem and sparse-graph codes (cont’d)

- Start with a small per-qubit error probability \( p \ll 1 \).
- Connect errors affecting common generators. For small \( p \) and a sparse code these form small disconnected clusters.
Threshold theorem and sparse-graph codes (cont’d)

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- Key observation: disconnected clusters can be detected independently; they do not affect each other’s syndromes.

This implies that errors formed by clusters of weight $w < d$ are all detectable
Threshold theorem and sparse-graph codes (cont’d)

- Start with a small per-qubit error probability $p \ll 1$.
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This implies that errors formed by clusters of weight $w < d$ are all detectable.

- Below percolation limit $p_c$, probability to have a cluster of large weight $w$ is exponentially small with $w$.
- Maximum cluster size grows logarithmically with $n$ (for small enough $p$ this is also true for confusing half-filled clusters)

Conclusion: as long as $d \propto n^\alpha$, $\alpha > 0$ (or even logarithmic), a sparse-graph code can correct errors at finite $p$. [Kovalev & LPP, ’13]
Percolation-based threshold for quantum LDPC codes

Actual value of the threshold for erasures: \( p_e \geq (z - 1)^{-1} \) for \((h, w)\)-limited code. For depolarizing channel:
\[
p_d \geq [2e(z - 1)]^{-2}
\] (assuming power-law distance).

Here \( z \equiv h(w - 1) \).

**Trouble:** This threshold is much weaker than what we have for the toric codes \((h = 2, w = 4)\), even though both thresholds are related to percolation.

**Reason:** This approximates code as a qubit-connectivity graph. Any structure associated with the action of generators is ignored.
Irreducible cluster counting algorithm

Definition 1 For a given stabilizer code, an undetectable operator is called irreducible if it cannot be decomposed as a product of two disjoint undetectable Pauli operators.

Algorithm for CSS code ($X$ errors)

- Order the stabilizer generators; pick a starting bit ($n$ choices)
- At each recursion step, deal with topmost ”unhappy” stabilizer generator and pick a bit among unselected points in its support (up to $w - 1$ choices)
- Recursion stops when syndrome is zero (an undetectable operator is found), or when there are no more positions for a given generator (have to go back).
- After $m$ recursion steps, return all irreducible undetectable operators of weight up to $m$. Complexity $N_m = n (w - 1)^{m-1}$.

This gives upper bound for the number $N_m$ of irreducible logical operators at $m \geq d$. 
Toric code example

Reducible cluster will be returned or not, depending on the order in which the numbered qubits are encountered.
Minimum-energy decoding

Let $P(E)$ be some error probability, energy $\varepsilon = -\ln P(E)$.

- For an (unknown) error $E$, let $E'$ be the minimum-energy error with the same syndrome $\Rightarrow E'E^\dagger$ is undetectable.
- Decompose $E'E^\dagger = \prod_j J_j$ into irreducible operators $J_j$.
- Error found correctly if $\varepsilon(J_j E) > \varepsilon(E)$ for all $J_j$ that are non-trivial logical operators ($J_j$ not in stabilizer)

Decoding is asymptotically correct at $n \to \infty$ if the probability for a ”bad” error for any irreducible $J \in C(S) \setminus S$ vanishes.

Let $\varepsilon(E)$ correspond to uniform uncorrelated errors. Then for a given $J$, probability $P_m$ of bad error only depends on $m \equiv \text{wgt } J$.

**Example:** Erasures with probability $p_e \Rightarrow P_m = p_e^m$.

Total probability to fail: $P_{\text{fail}} \leq \sum_{m \geq d} P_m N_m \leq \frac{n[(w-1)p_e]^d}{1 - (w-1)p_e}$
Improved cluster counting

For toric code, $w = 4$, and this bound is the same as simple-minded walk counting ($N_m \sim n 3^{m-1}$)

Power-law scaling of $N_m$ for different codes — exponents can be used for improved bounds, just like SAW exponent in the case of the toric code [$\zeta_6 \approx 4.76$, $\zeta_7 \approx 5.74$, $\zeta_8 \approx 5.79$ and $\zeta_9 \approx 6.78$]
Combination of erasures and independent $X/Z$ errors

Combined erasures (probability $p_e$) and $X$ errors (probability $p$).

Probability of $E$: $a$ erasures and $b$ $X$ errors in a cluster of size $m$:

$$P_E = \binom{m}{a} p_e^a (1 - p_e)^{m-a} \binom{m-a}{b} p^b (1 - p)^{m-a-b}.$$ 

Probability of $JE$ (invert bits outside of the erasure):

$$P_{JE} = \binom{m}{a} p_e^a (1 - p_e)^{m-a} \binom{m-a}{b} (1 - p)^b p^{m-a-b}.$$ 

Bad errors: $P_E \leq P_{JE}$, which gives $m - a - 2b > 0$.

Upper bound for bad error probability in a cluster of size $m$:

$$P_m = \left\{ p_e + (1 - p_e) [4p(1 - p)]^{1/2} \right\}^m.$$ 

With code distance scaling as a power law $d \geq An^\alpha$, $\alpha > 0$, minimum-energy decoding asymptotically successful if

$$p_e + (1 - p_e) [4p(1 - p)]^{1/2} \leq (w - 1)^{-1}.$$
Fault-tolerant case

With syndrome errors, use aux 3D code with CSS-like generators (analog of 3D line matching):

\[ P = \left( I_m \otimes H_{r \times n}, R_{m \times (m-1)} \otimes I_r \right) \]

Degeneracy generator:

\[ Q = \left( \begin{array}{ccc}
[RT]_{(m-1) \times m} \otimes I_n & I_{m-1} \otimes [HT]_{n \times r} \\
I_m \otimes G'_{r' \times n}, & 0
\end{array} \right) \]

Repetition code check matrix:

\[ [RT]_{(m-1) \times m} \equiv \begin{pmatrix}
1 & 1 \\
0 & 1 & 1 \\
& & \ddots & \ddots \\
& & & 1 & 1
\end{pmatrix} \]

Bound the number of clusters of size \( m \), with \( m_q \) qubit errors:

\[ \overline{N}_{m,m_q} \leq \binom{m}{m_q} w^{m_q} 2^{m-m_q} \]

For combination of uncorrelated erasures \( (p_e) \), depolarizing \( (p) \), and syndrome errors, with distance \( d \geq D \ln n \), we get

\[ 4[q(1-q)]^{1/2} + wyY \leq e^{-1/D}, \]

\[ Y \equiv p_e + (1-p_e) \left\{ \frac{2p}{3} + 2 \left[ \frac{p}{3}(1-p) \right]^{1/2} \right\} \]
Summary

- New analytic lower bound for the thresholds with minimum-energy decoder
  - Same accuracy as counting SAWs for the toric code
  - Simple expressions for uncorrelated errors
  - Phenomenological syndrome errors included on equal footing
  - Way better than the old percolation-based bound

- Erasure threshold, e.g., $p_e \geq (w - 1)^{-1}$ for CSS codes, also gives bounds:
  - for code rate, using $1 - R \geq 2p_e \Rightarrow R < 1 - 2/(w - 1)$
  - for codes with transverse logical ops in $m$th level of Clifford hierarchy, $p_e \leq 1/w$ [Yoshida & Pastawski (2014)]

- This corresponds to a bound on percolation of (binary) cycles on hypergraphs

- Yet percolation on a graph (like the old bound) can be also used:
  - With large variations of $w$, e.g., $p_e \geq 1/\lambda_{\text{max}}(A)$ [Hamilton & LPP, 2014]
  - With correlated errors [in progress]

  Not clear if something similar can be done in the present case.

  **Need to come up with MF theory for percolation on hypergraphs**

A good postdoc is needed to work on this, LDPC codes & related stat-mech!