Fibre bundle framework for unitary quantum fault tolerance

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December 18, 2014

Joint work with Daniel Gottesman, arXiv:1309.7062

Motivations

- Fault tolerance $\rightarrow$ robust computer (major obstacle):
  - Classical fault tolerance – e.g. repetition code
  - Quantum fault tolerance – e.g. transversal gates with ancilla constructions, topological fault tolerance
- We know of various protocols of fault tolerance, we want to understand them in some unified framework.
- Achieved:
  - Developed conjecture of a *global* and *geometric* picture of unitary quantum fault tolerance.
  - Proof of conjecture for transversal gates
  - Proof of conjecture for a family of topological codes, including the toric code
- Hope: new insights, new fault tolerant protocols ...

Main idea

Conjecture

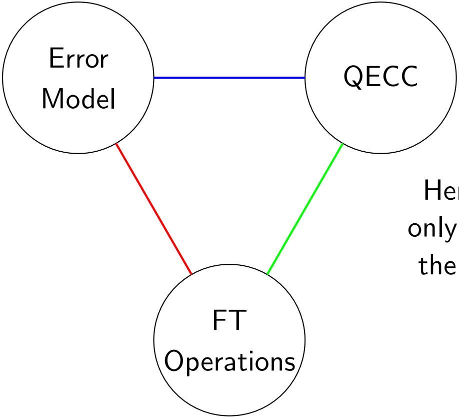
Correspondence for appropriate fibre bundles F, with base space M:

Unitary fault tolerance	$\longleftrightarrow$	*Fibre bundle F with flat proj. connection*
Fault-tolerant logical gates	$\longleftrightarrow$	*Monodromy rep. of $\pi_1(M)$*

The conjecture ($\rightarrow$) is proven for the cases of: focus of the talk

- transversal gates and
- generalized string operators for a family of topological codes.

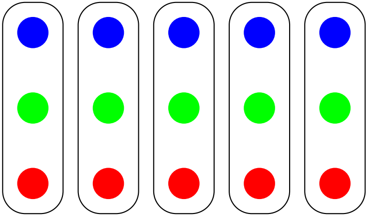
Ingredients of a fault-tolerant protocol



Here, we focus on only the QECCs and the FT operations.

Example 1: Transversal gates definition

- Code blocks (of equal size): qudits represented by same colour
- Transversal gates: Interact the i^{th} qudit of each block

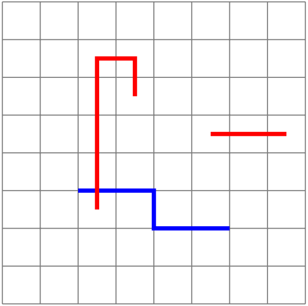


A transversal gate on multiple blocks of a QECC can be considered as a transversal gate on a single block of a QECC with larger physical qudits. We group together qudits in the same column to make the larger qudits.

Example 2: Modified toric codes and String operators

- Original toric code by Kitaev in arXiv:quant-ph/9707021
- Modified toric code Hamiltonian (primal defects at S_v, dual at S_f):

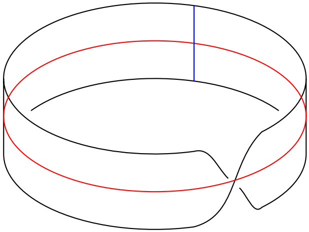
$$H(S_v, S_f) = -\sum_{v \in V \setminus S_v} A_v - \sum_{f \in F \setminus S_f} B_f + \sum_{v \in S_v} A_v + \sum_{f \in S_f} B_f .$$



String operators transport defects.

Fibre bundle – The Möbius band

- Constituents: total space, base space, fibre, structure group
- An example:



A nontrivial fibre bundle over the base space S^1 (in red) with fiber $\mathbb{R}$ (fiber at one point shown in blue). Structure group is $\mathbb{Z}_2$ in this case.

Base space: Codes as the Grassmannian manifold

Over the next couple of slides, we build up the "big vector bundle" for our picture, from mathematical objects natural for QEC. First,

- Base space is the Grassmannian (a set of codes):
  - An $((n, K))$ qudit code is a K-dimensional subspace in $\mathbb{C}^N$ where $N = d^n$ (n-qudit Hilbert space).
  - $\mathrm{Gr}(K, N) = \{\text{The set of } K\text{-dimensional subspaces in } \mathbb{C}^N\}$
    - Example: $\mathbb{C}P^1 = \mathrm{Gr}(1, 2)$
    - Known as the *Grassmannian*.
  - Clearly, for $N = d^n$,

$$\mathrm{Gr}(K, N) = \{\text{The set of } ((n, K)) \text{ qudit codes}\}.$$

Vector bundle: Codewords as the tautological vector bundle $\xi(K, N)$

- Total space is the tautological vector bundle (a set of codewords):
  - A codeword in an $((n, K))$ qudit code is a pair (C, w) where C is an $((n, K))$ qudit code and $w \in C$ is a vector.
  - $\xi(K, N)$ is a vector bundle with:
    - Base space is $\mathrm{Gr}(K, N)$, consisting of subspaces W
    - Fibre over W is W itself, i.e. the elements are vectors $w \in W$
    - Known as the *tautological vector bundle*
  - Similarly, for $N = d^n$, we have the natural mathematical-QEC correspondence:

$$\xi(K, N) = \{\text{Codewords in some } ((n, K)) \text{ qudit code}\}. \quad (1)$$

Some correspondences between the theory of QECCs and that of fibre bundles

A summary:

Quantum information objects	Mathematical objects
Space of $((n, K))$ qudit codes	Grassmannian $\mathrm{Gr}(K, N)$ where $N = d^n$
Space of the *codewords* (C, w)	tautological vector bundle $\xi(K, N)$
Space of the *encodings* or orthonormal K-frames β in $\mathbb{C}^N$	tautological principle $\mathcal{U}(K)$-bundle $P(K, N)$

Dynamics in unitary fault tolerance (or unitary QM)

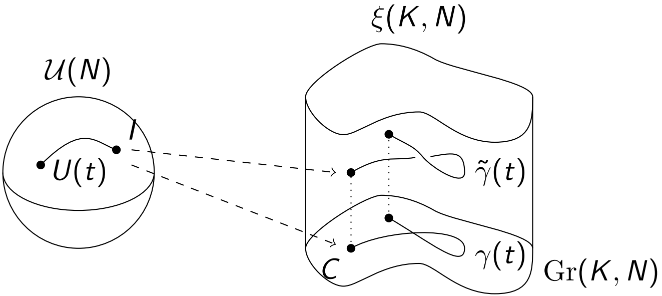
Definition

A *unitary evolution* is a one-parameter family $U(t)$ of unitary operators such that, at time 0, $U(0) = I$, and as time passes, $U(t)$ evolves smoothly (or piecewise smoothly) with time, until at time 1, it accomplishes some target unitary $U(1) = U$.

- Modelling unitary evolutions in our geometric picture
  - Task 1: Unitary evolutions of the codewords (states)
  - Task 2: Unitary evolutions of the QECC (subspaces)

"Dynamics" in the "big vector bundle"

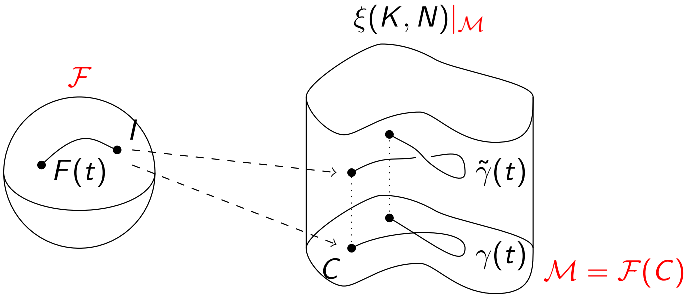
- Given a unitary evolution $U(t)$ and a code C, we obtain:
  - a path in the bundle (evolution of codewords)
  - a path in the base space (evolution of codes)



- Resembles a parallel transport/connection (*pre-connection*)
  - Problem: The lift $\tilde{\gamma}(t)$ of $\gamma(t)$ might not be unique.

Restricting bundle to $\mathcal{M} \subset \mathrm{Gr}(K, N)$ and $\mathcal{F} \subset \mathcal{U}(N)$

Schematic illustration of the restrictions:



Conjecture (fault tolerance magic)

For appropriate restrictions (depending on FT protocol), FT $\Rightarrow$ the natural (proj.) pre-connection becomes an flat (proj.) connection.

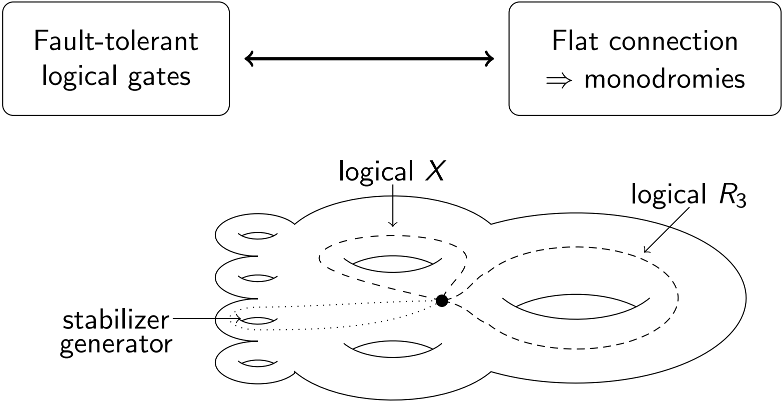
Examples: $\mathcal{F}$ and $\mathcal{M}$

- Example 1: Distance ≥ 2 code with transversal gates
  - C any code with distance ≥ 2.
  - $\mathcal{F} = \{\text{Transversal gates}\} \subset \mathcal{U}(N)$
  - $\mathcal{M} = \mathcal{F}(C) \subset \mathrm{Gr}(K,N)$
  - Flatness results follow from arXiv:0811.4262 (Eastin and Knill)
- Example 2: Toric code with string operators

$$\begin{array}{ccccccc} \mathcal{F}_{\mathrm{discr}} \circlearrowright & & \mathcal{F}_{\mathrm{graph}} \circlearrowright & & \mathcal{F}_{\mathrm{ext}} \circlearrowright & & \\ C_K^{HC,(n_v,n_f)} & \subset & \mathring{\mathcal{M}} & \subset & \mathcal{M} & \subset & \mathrm{Gr}(K,N) \end{array}$$

  - $\mathcal{M} \cong$ defect configuration space (fixed number of defects, hardcore condition); There is freedom in the choice of $\mathcal{M}$.
  - Flatness results in arXiv:1309.7062 (Gottesman and Zhang)

Corollary: Monodromy action



A cartoon of $\mathcal{M}$ for single-block transversal gates for the 5-qubit code.

Summary and Future work

- Imperfection of the current conjecture:
  - Multiple valid choices of $\mathcal{M}$ for the same protocol
  - Lacks concrete instructions to construct $\mathcal{M}$
- Improve conjecture: incorporate error model, propose *canonical* construction of $\mathcal{M}$ for each fault-tolerant protocol.
  - Stricter correspondence between FT protocols (with error models etc.) and fibre bundles with flat (proj.) connection
  - Will enable us to read off new FT protocols from a "nice" bundle construction with flat (proj.) connection
  - Proof of improved conjecture
- Extend to *full* fault tolerance: e.g. ancilla constructions (appending extra degrees of freedom), measurements
- Other applications of the this geometric picture, e.g. TQFTs and topological phases
- Thank you!