A combinatorial application of quantum information in percolation theory

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From percolation to topological codes

- Quantum erasure channel: Each qubit is erased (lost) with probability p, independently.

- Relation with percolation: For Kitaev's toric code, correction of erasures is related with a statistical mechanical model called percolation.

- Application: Apply results from percolation theory to surface codes.
  (Stace, Barrett Doherty - 2009)

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Goal: derive results in percolation from quantum information.

Overview

Percolation Theory

From percolation to quantum error correction

Three bounds on the threshold

- no-cloning bound
- LDPC codes bound
- homological bound

Why percolation?

The melting of ice is a phase transition at the critical point $T = 0°C$: There is a discontinuous evolution of macroscopic properties of water.

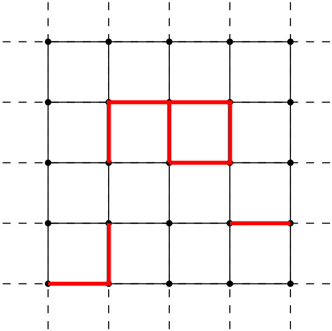
Question:

How do local interactions between particles induce a global behaviour?

Why percolation? It is perhaps the simplest model which exhibits a phase transition.

Percolation in $\mathbb{Z}^2$

Each edge is red, independantly with probabily p.



Question: is there an infinite red component ?

Percolation in $\mathbb{Z}^2$

There is a phase transition at p_c:

- if $p < p_c$, there is an infinite red component with proba 0,
- if $p > p_c$, there is an infinite red component with proba 1.

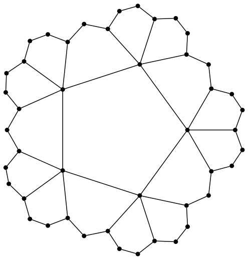
Goal: Determine the value of p_c.

Theorem (H. Kesten, 1980 - conjectured 20 years before)

In the square lattice we have: $p_c = 1/2$.

Percolation in hyperbolic lattices

Let $G(m)$ be the m-regular planar tiling.



- The exact value of p_c is unknown.
- The numerical estimation of p_c is difficult.

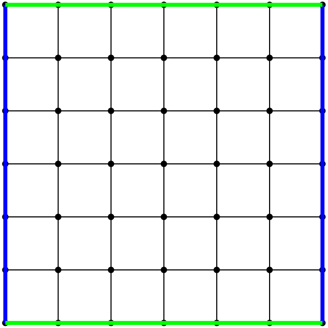
(Benjamini, Schramm, and later Baek, Kim, Minnhagen and Gu, Ziff)

We will use quantum information theory to bound p_c.

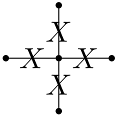
From percolation to topological codes

Kitaev's toric codes (Kitaev - 1997)

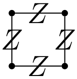
- Place a qubit on each edge of a torus.
- This gives a global state $|\psi\rangle \in (\mathbb{C}^2)^{\otimes n}$ with $n = |E|$.



site operator $X_v =$



face operator $Z_f =$

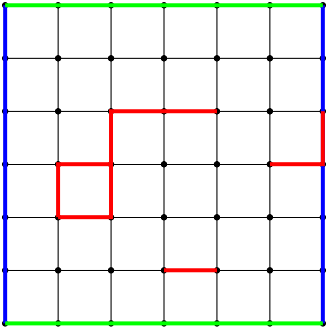


The toric code is the ground space of

$$H = -\sum_v X_v - \sum_f Z_f$$

A problematic erasure

Each qubit is erased (lost), independently, with probability p.



Correctable $\Leftrightarrow$ erased clusters are planar

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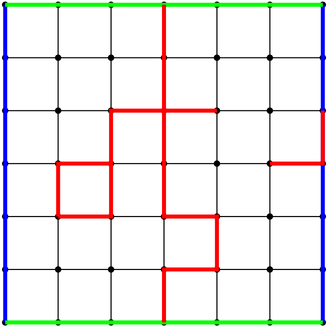

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From percolation to toric codes

For large tilings, we have:

Uncorrectable erasures $\approx$ Infinite clusters in percolation

Threshold for percolation in $\mathbb{Z}^2 \Longrightarrow$ Threshold for toric codes:

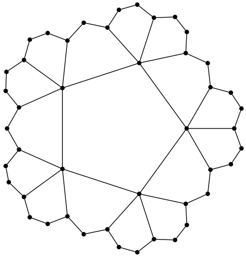
- $p < p_c \Rightarrow$ the toric code has a good performance

(Stace, Barrett Doherty - 2009)

Construction of hyperbolic codes

First step: Relate hyperbolic percolation to topological codes.

Using finite versions of $G(m)$, we can define hyperbolic codes: (Freedman, Meyer, Luo - 2001, Zémor 2009)



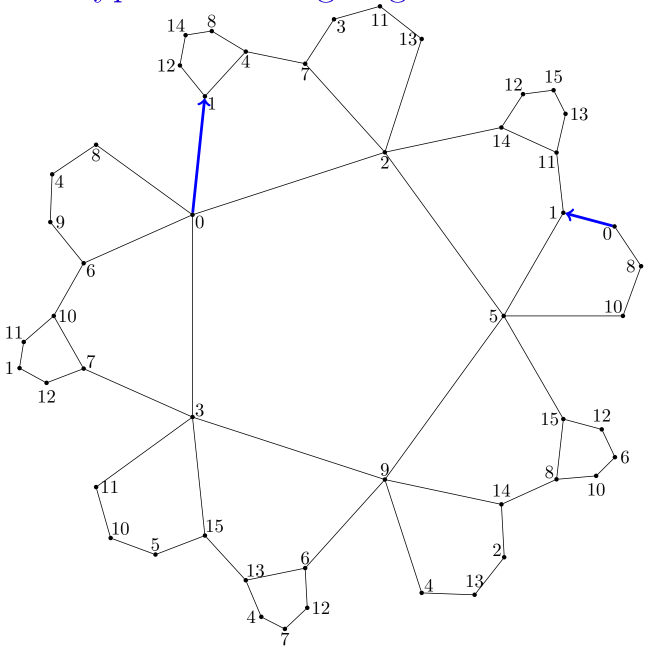
Place a qubit on each edge, then

- Plaquette operators X_v correspond to the edges incident to a vertex
- Site operators Z_f correspond to faces.

The hyperbolic code is the ground space of

$$H = -\sum_v X_v - \sum_f Z_f.$$

A finite hyperbolic tiling of genus 5



From percolation to hyperbolic codes

We use quotients of $G(m)$ (Proposed by Siran '01) such that

- $G_r(m)$ is a finite graph
- $G_r(m)$ locally looks like $G(m)$
  (balls of radius r are planar)

Then, for large r, we have:

$p < p_c(G(m)) \Rightarrow$ hyperbolic codes have a good performance.

Application to percolation

- No-cloning bound

Capacity of the quantum erasure channel

$$\begin{array}{ccccccc} k \text{ qubits} & & n \text{ qubits} & & n \text{ qubits} & & k \text{ qubits} \\ \text{m} & \longrightarrow \boxed{\text{Encoding}} \longrightarrow & \text{x} & \longrightarrow \boxed{\text{Channel}} \longrightarrow & \text{x'} & \longrightarrow \boxed{\text{Decoding}} \longrightarrow & \text{m'} \end{array}$$

What is the highest rate $R = k/n$ with $P_{err} \to 0$?
$\longrightarrow$ It is the capacity of the channel.

Theorem (Bennet, DiVicenzo, Smolin - 97)

The capacity of the quantum erasure channel is $1 - 2p$.

Derived from the no-cloning theorem.

A no-cloning upper bound in percolation

Main argument: if $p < p_c$ then $R = 1 - \frac{4}{m} \leq 1 - 2p$

Theorem (D., Zémor - ITW 10)

The critical probability on the graph $G(m)$ satisfies:

$$p_c \leq \frac{2}{m}.$$

Easy combinatorial bounds:

$$\frac{1}{m-1} \leq p_c \leq 1 - \frac{1}{m-1}.$$

Application to percolation

- No-cloning bound
- LDPC bound

Improving the no-cloning bound

- The no-cloning bound is tight only if hyperbolic codes achieve capacity.
- Hyperbolic quantum codes are defined by bounded weight generators. (LDPC).
- Classical intuition: Classical LDPC codes cannot achieve the capacity.

Difficulty: the no-cloning bound is not related with the codes.

A combinatorial bound

$$\mathbf{H} = \begin{pmatrix} I & X & Z & Y & Z \\ Z & Z & X & I & Z \\ I & Y & Y & Y & Z \end{pmatrix}$$

$$\mathcal{E} = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \end{pmatrix}$$

- There are 4^2 errors $E \subset \mathcal{E}$

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Lemma

We can correct $2^{\operatorname{rank}\mathbf{H} - (\operatorname{rank}\mathbf{H}_{\bar{\mathcal{E}}} - \operatorname{rank}\mathbf{H}_{\mathcal{E}})}$ *errors* $E \subset \mathcal{E}$.

Combinatorial version of the no-cloning bound

Let $(\mathbf{H}_t)$ be a sequence of stabilizer matrices of codes of rate R.

Theorem (D., Zémor - QIC 2013)

If $P_{err} \to 0$ then

$$R \leq 1 - 2p - D(p),$$

where

$$D(p) = \limsup_t \frac{\mathbb{E}_p[\operatorname{rank} \mathbf{H}_{t,\bar{\mathcal{E}}} - \operatorname{rank} \mathbf{H}_{t,\mathcal{E}}]}{n_t}.$$

Corollary: When $p \leq 1/2$, we have $R \leq 1 - 2p$.

Remark: With hyperbolic codes, the matrices $\mathbf{H}_t$ are sparse.

Rank of a random sparse matrix

$$\left(\begin{array}{c|c} \underbrace{\quad \mathbf{H}_{\mathcal{E}} \quad}_{pn \text{ columns}} & \qquad\qquad\qquad\qquad\qquad\qquad \end{array}\right)$$

- Typically: $\mathbf{H}_{\mathcal{E}}$ is a $r \times np$ matrix

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- BUT for a sparse matrix $\mathbf{H}$, there are αn null rows in $\mathbf{H}_{\mathcal{E}}$
  $\longrightarrow$ Bound on $D(p)$.

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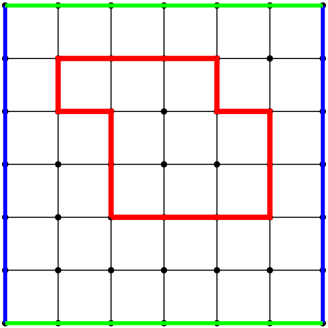
- BUT for a sparse matrix $\mathbf{H}$, there are αn null rows in $\mathbf{H}_{\mathcal{E}}$ $\longrightarrow$ Bound on $D(p)$.
- Similarly, there are βn identical rows of weight 1 ... $\longrightarrow$ more accurate bound.

Application to percolation

- No-cloning bound
- LDPC bound
- Homological bound

Homology of the torus

Goal: Remove the quantumness.



$H_1(G)$ = Homology group = cycles up to faces.

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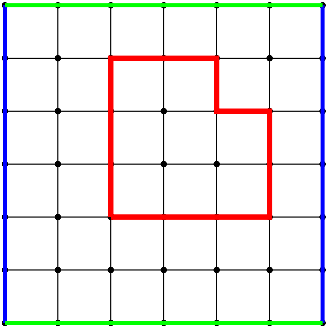

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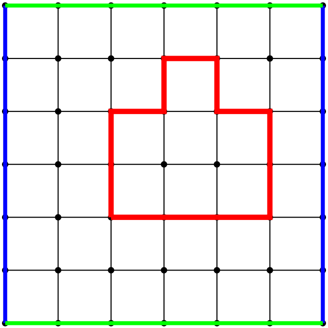

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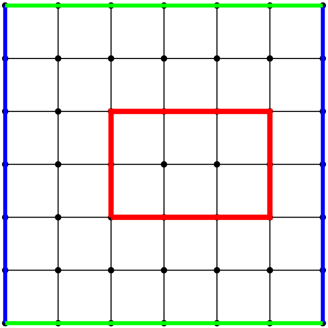

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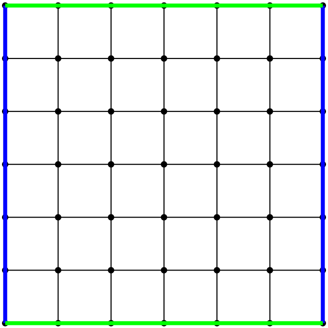

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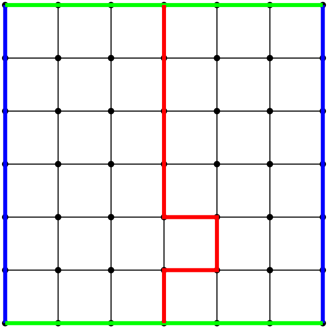

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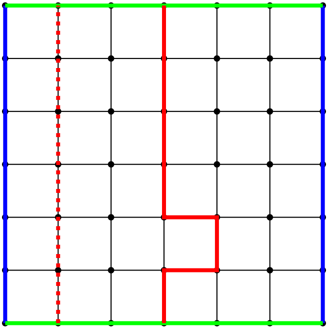

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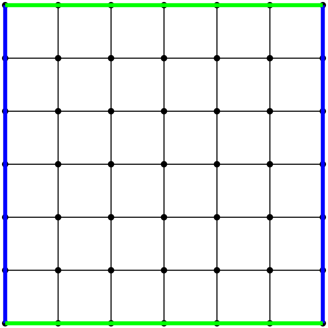

$H_1(G)$ = Homology group = cycles up to faces.

$$H_1(G) = \langle \gamma_{\text{horizontal}}, \gamma_{\text{vertical}} \rangle$$

Intuition: threshold for appearance of homology

Recall that correctable erasure $\Leftrightarrow$ no homology

- Let G_r be the finite version of $G(m)$.
- Let $G_{r,p}$ be a random subgraph of G_r.

Basic idea of our homological bound:

1. If $p < p_c$, the dimension of $H_1(G_{r,p})$ is small.
2. Compute the expected dimension $\mathbb{E}(\dim H_1(G_{r,p}))$.

Then, if $\mathbb{E}(\dim H_1(G_{r,p}))$ is large, we are beyond p_c.

A functional equation below p_c

Theorem (D. Zémor - 2014)

If $p < p_c(G(m))$ *then*

$$p - \frac{2}{m} + D(p) = 0.$$

Where

$$D(p) = \limsup_r \mathbb{E}_p \left(\frac{\operatorname{rank} G^*_{r,1-p} - \operatorname{rank} G_{r,p}}{|E_r|} \right).$$

Computation of $D(p)$

By combinatorial arguments, we obtain $D(p)$ as a function of the subgraphs of $G(m)$.

Theorem (D., Zémor - 2014)

$D(p)$ is equal to

$$\frac{2}{m} \sum_{C \in \mathcal{C}(v)} \left(\frac{1}{|V(C)|} \left(p^{|E(C)|}(1-p)^{|\partial(C)|} - (1-p)^{|E(C)|} p^{|\partial(C)|} \right) \right),$$

where $\mathcal{C}(v)$ denotes the set of connected subgraphs C of $G(m)$ containing a fixed vertex v.

$C = \{v\} \Rightarrow \frac{2}{m}((1-p)^5 - p^5)$.
$C = \{v, w\} \Rightarrow \frac{2}{m}\frac{1}{2}(p^1(1-p)^8 - p^8(1-p)^1)$.

Numerical results in G(5)

- Simple bounds: $\frac{1}{m-1} \leq p_c \leq 1 - \frac{1}{m-1}$, thus

$$0.25 \leq p_c \leq 0.75$$

- No-cloning bound (D., Zémor - 2010):

$$p_c \leq 0.40$$

- "Monte Carlo upper bound" (Gu, Ziff - 2012):

$$p_c \lesssim 0.34$$

- Matricial bound (D. Zémor - 2013):

$$p_c \lesssim 0.38$$

- Lyons remark: Benjamini, Shramm '96 + Haggstrom, Jonasson, Lyons '02

$$p_c \lesssim 0.31$$

- Homological bound (D., Zémor - 2014):

$$p_c \leq 0.2999...$$

Conclusion

Results:

- It is a purely combinatorial application of quantum information.
- The critical probability is local.
- Feedback on hyperbolic codes: precise upper bound on the threshold.

Open questions:

- Lower bound on p_c.
- Case of non self-dual hyperbolic tilings.
- We conjecture that our homological bound is tight.
- Recover Kesten's result

Thank you for your attention!