Finite Length — The Final Frontier

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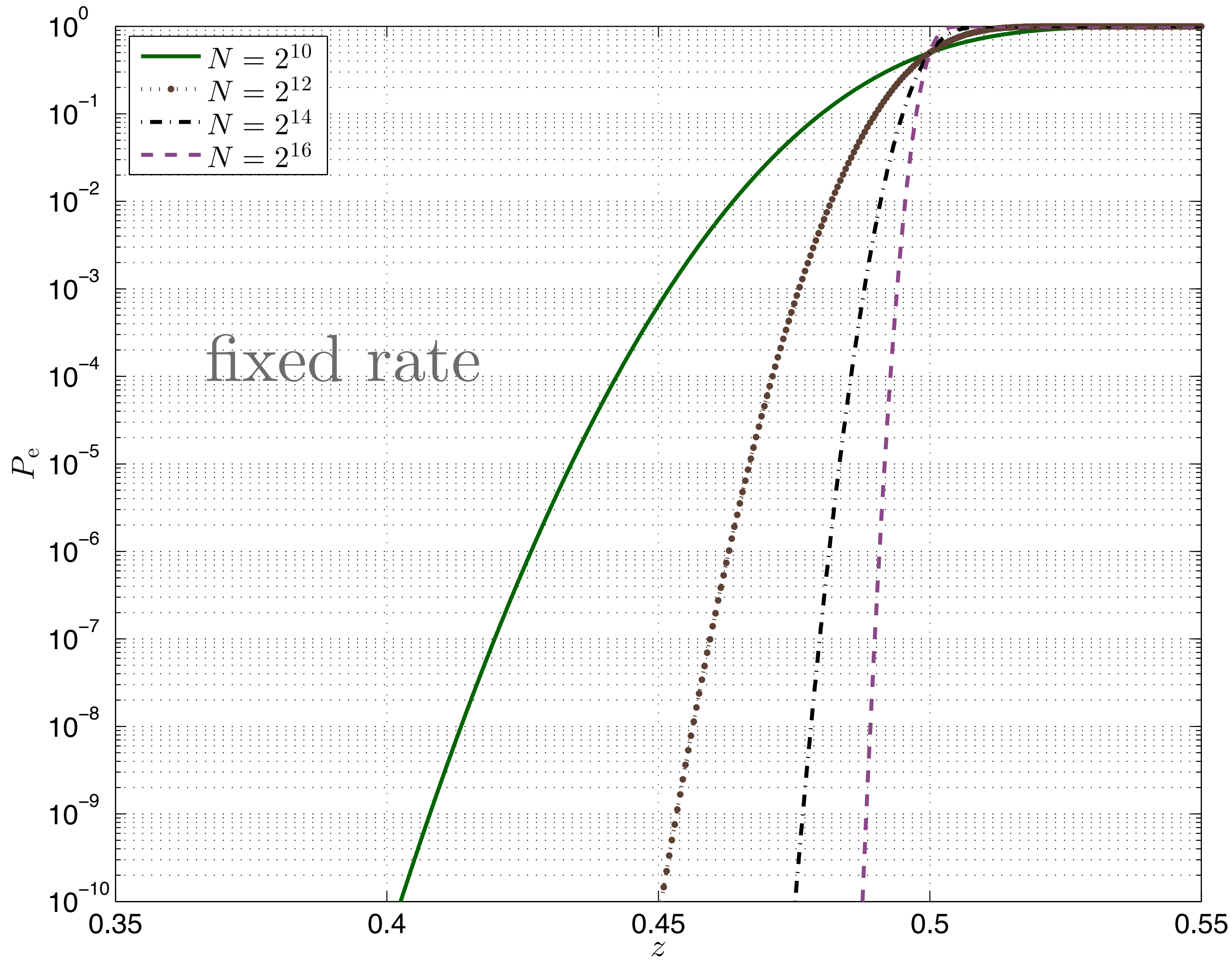
EPFL

Based on joint work with Hamed Hassani and Marco Mondelli.

December 17th, 2014

ÉCOLE POLYTECHNIQUE
FÉDÉRALE DE LAUSANNE

family of codes of increasing N

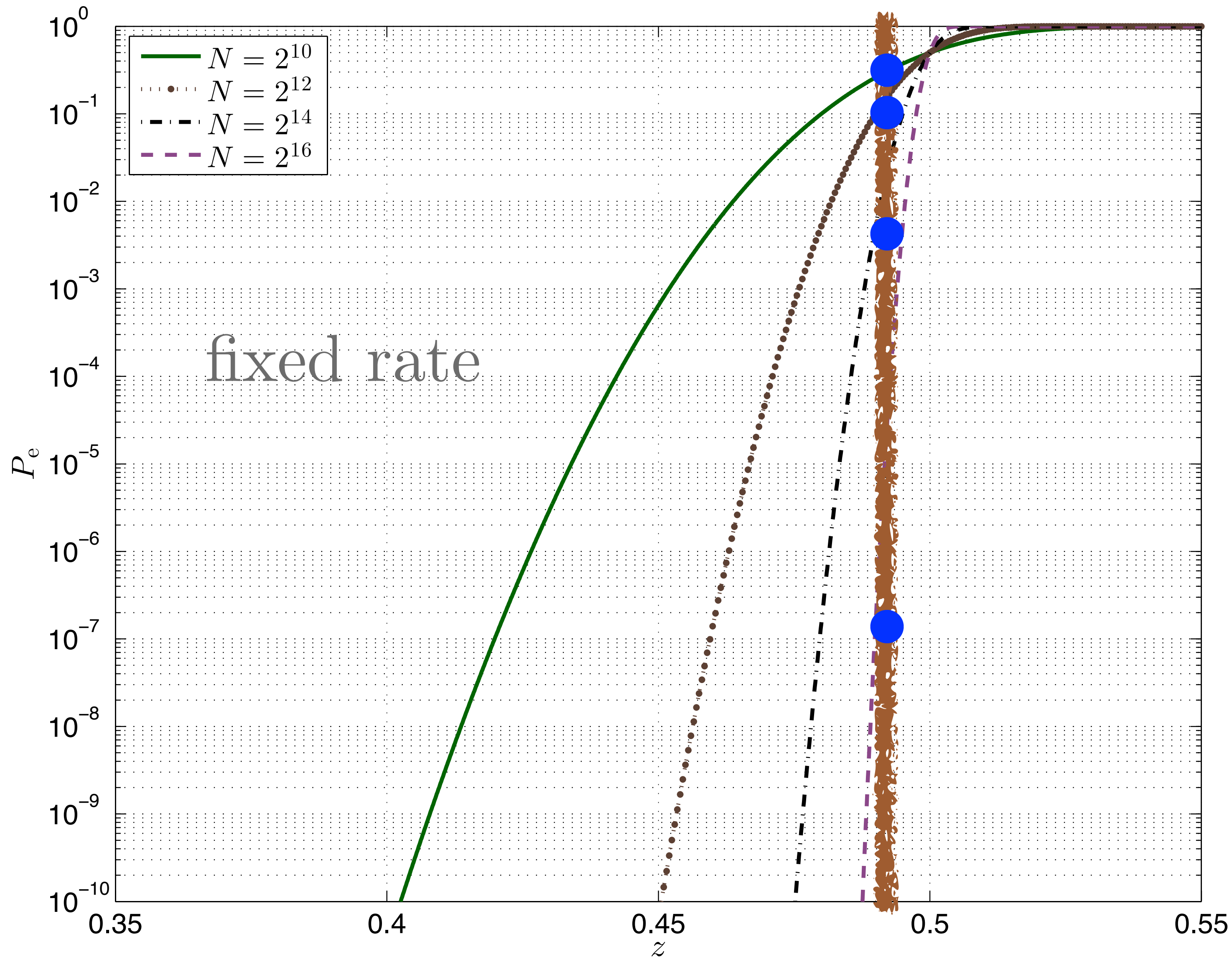


good channel

bad channel

Error Exponent

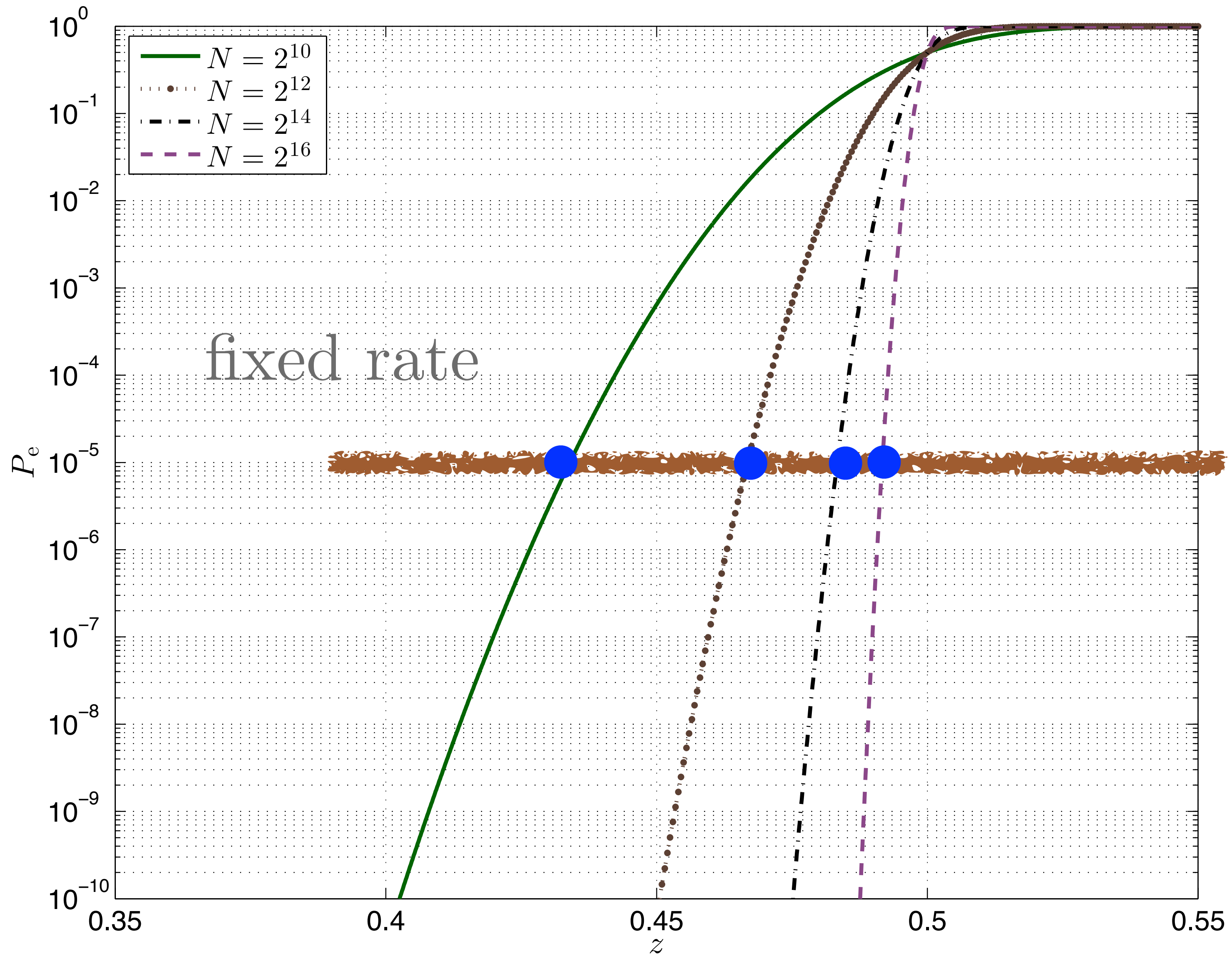
family of codes of increasing N



good channel bad channel

Scaling Exponent

family of codes of increasing N



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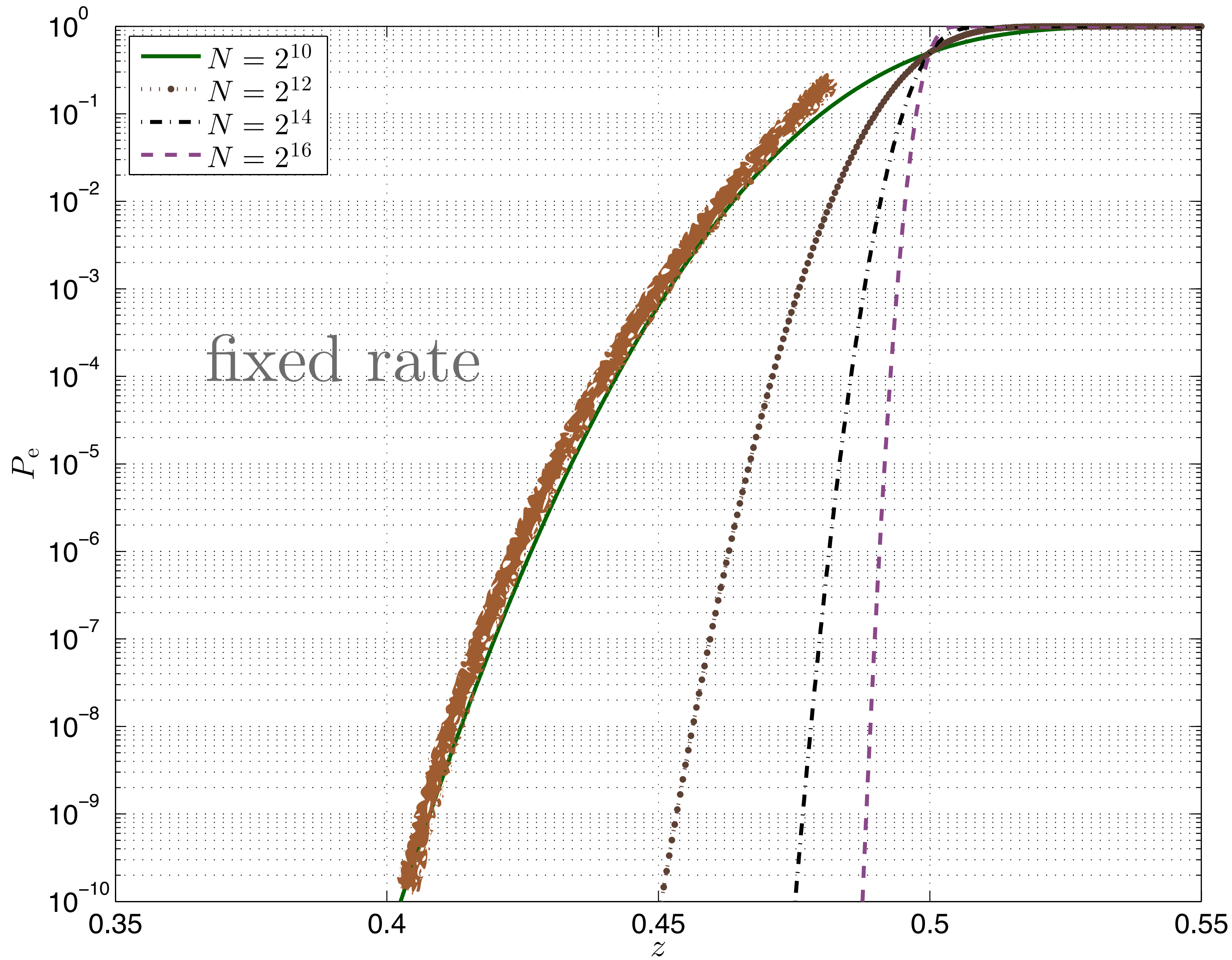

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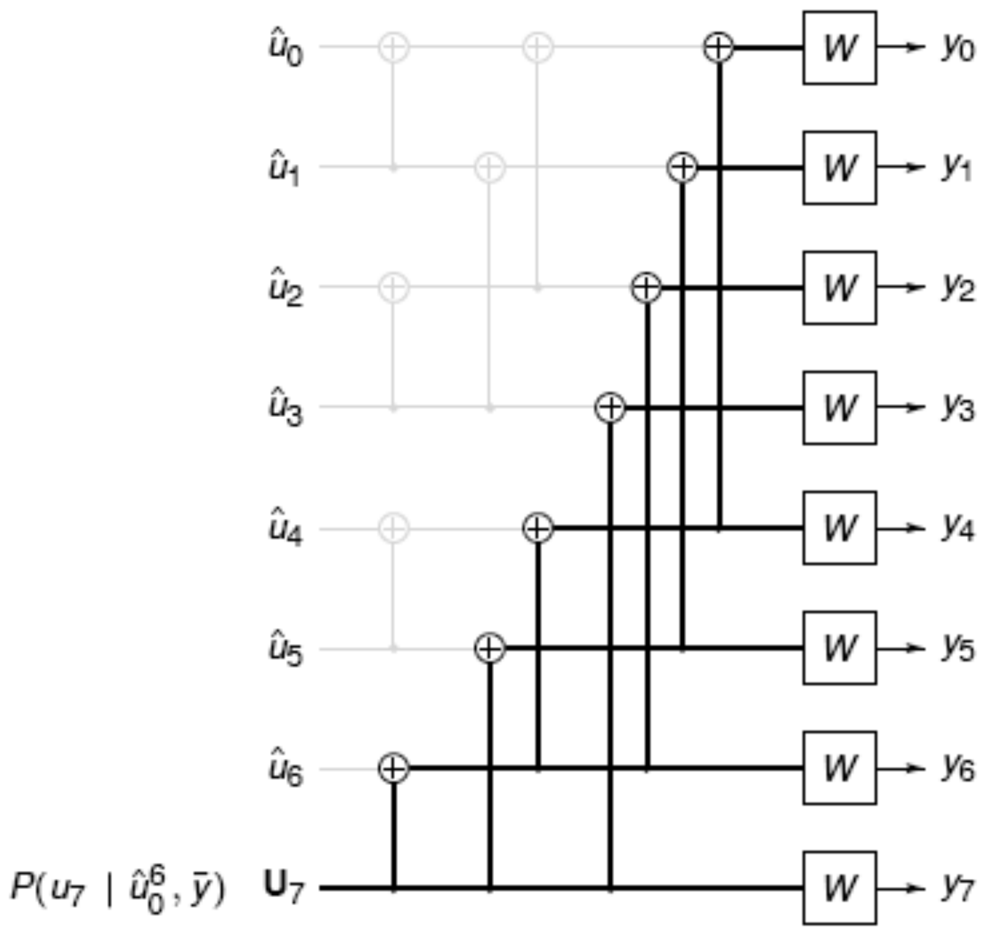
Polar

[Arikan 2007]

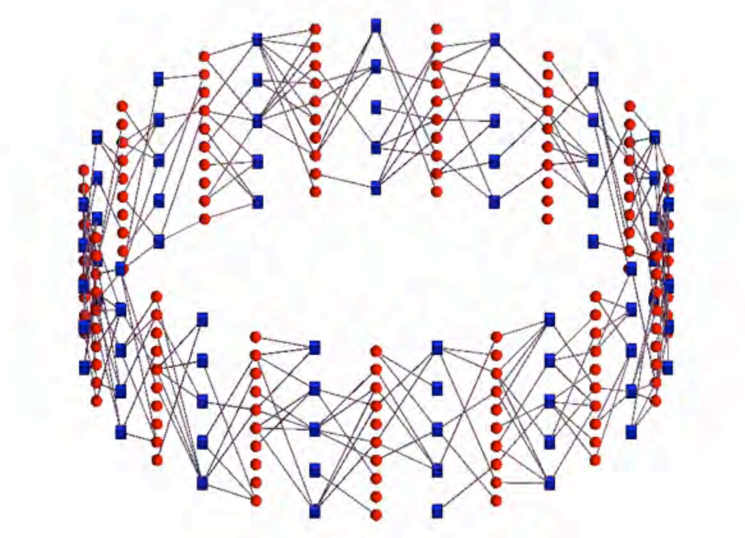


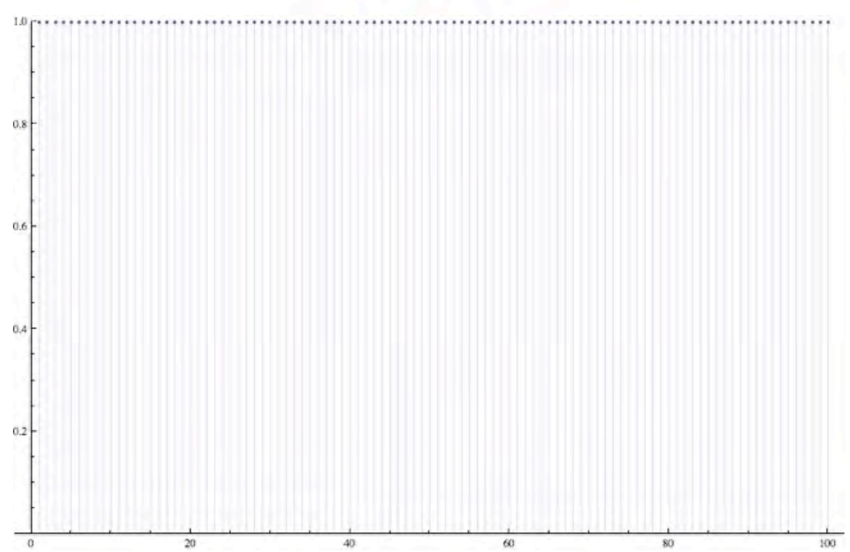
capacity achieving on BMS channels
very elegant and short proof
efficient construction
low complexity (log(n) operations per bit)
low error floors
predictable performance
many applications

not universal
large blocklengths needed

Spatially Coupled Codes

[Felstroem and Zigangirov 99, Convolutional LDPC Codes]





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proof more involved
no simple bounds for finite lengths

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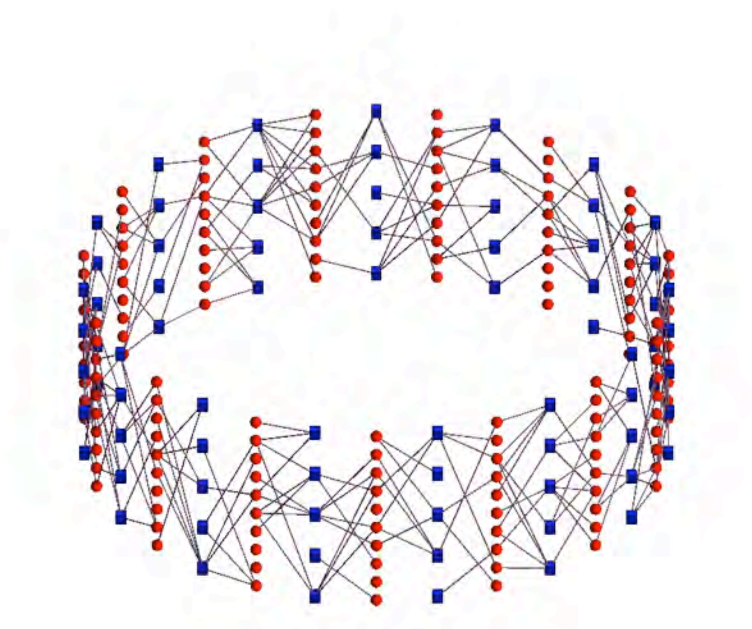

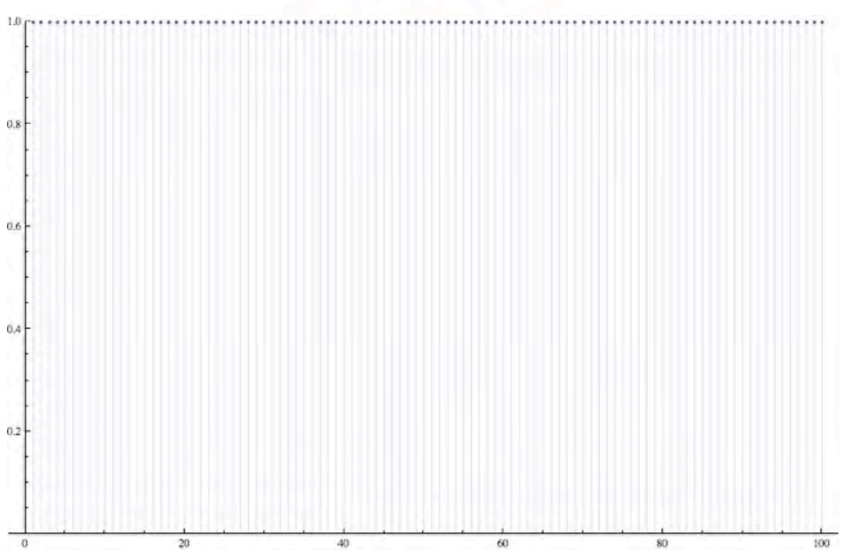

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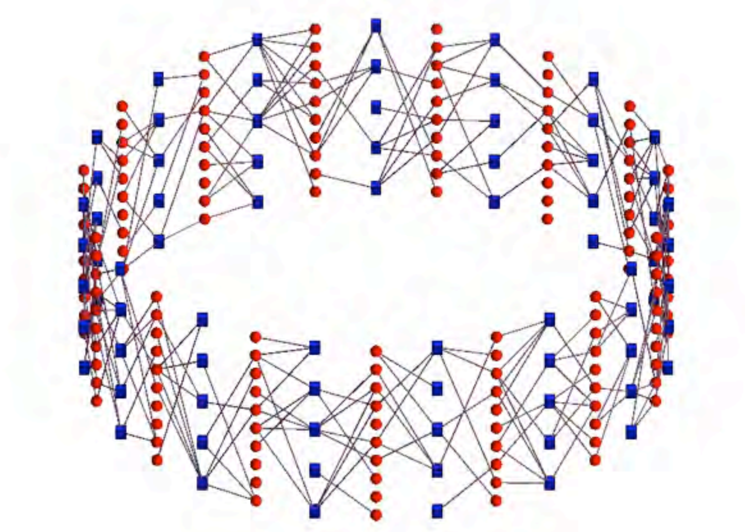

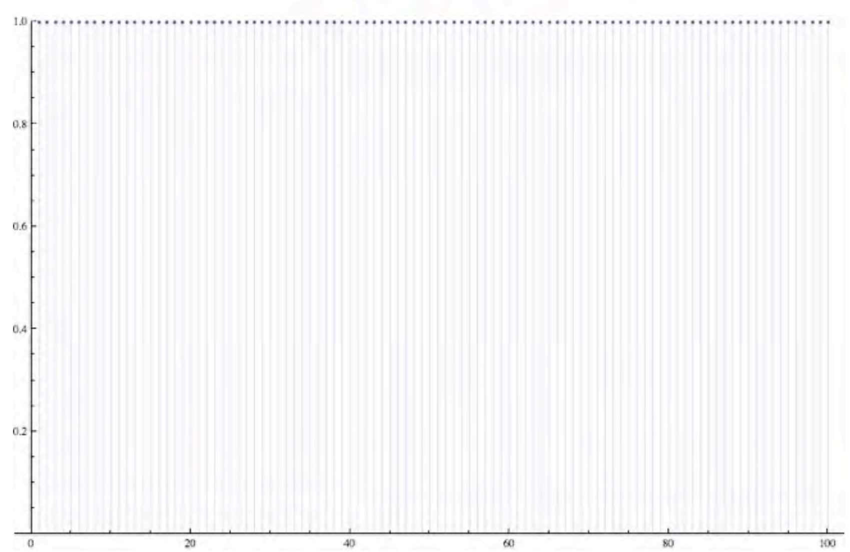

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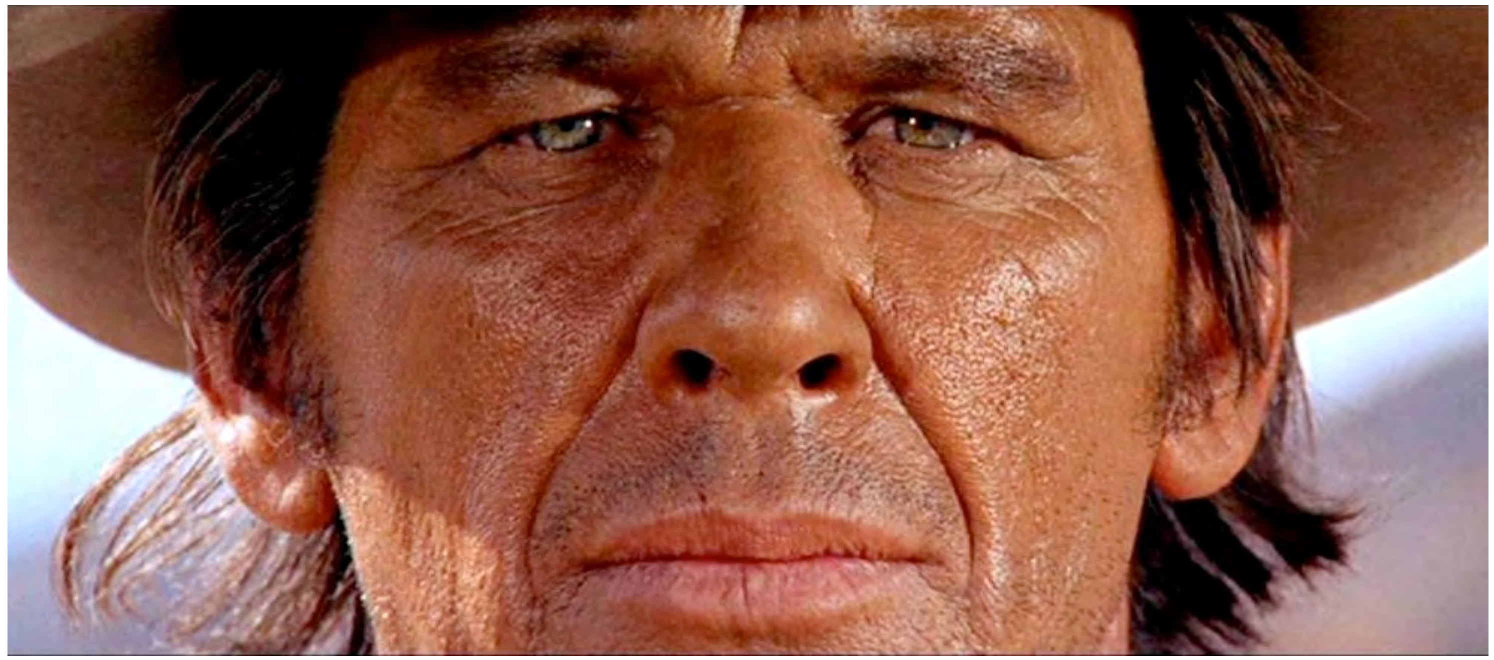
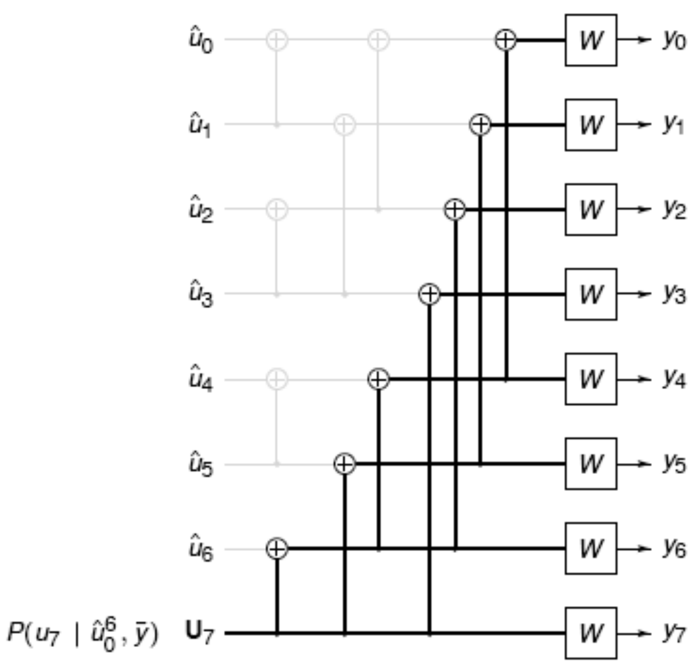
Showdown




$\hat{u}_0$
$\hat{u}_1$
$\hat{u}_2$
$\hat{u}_3$
$\hat{u}_4$
$\hat{u}_5$
$\hat{u}_6$
$P(u_7 \mid \hat{u}_0^6, \bar{y})$ $\mathbf{U}_7$
W
y_0
y_1
y_2
y_3
y_4
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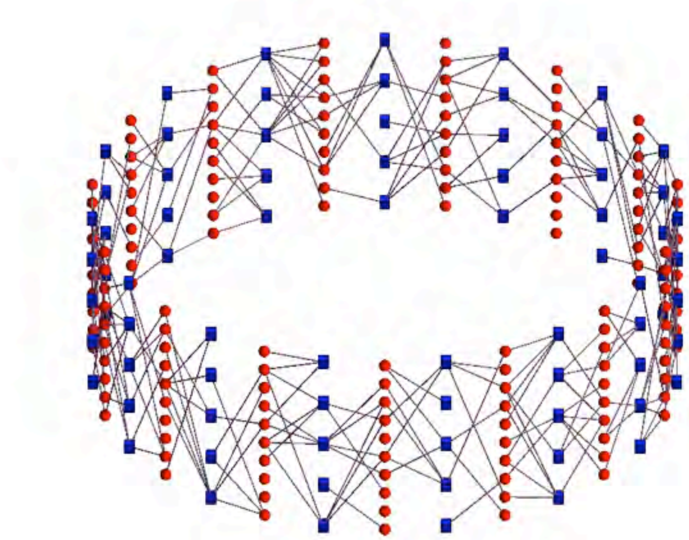
y_6
y_7



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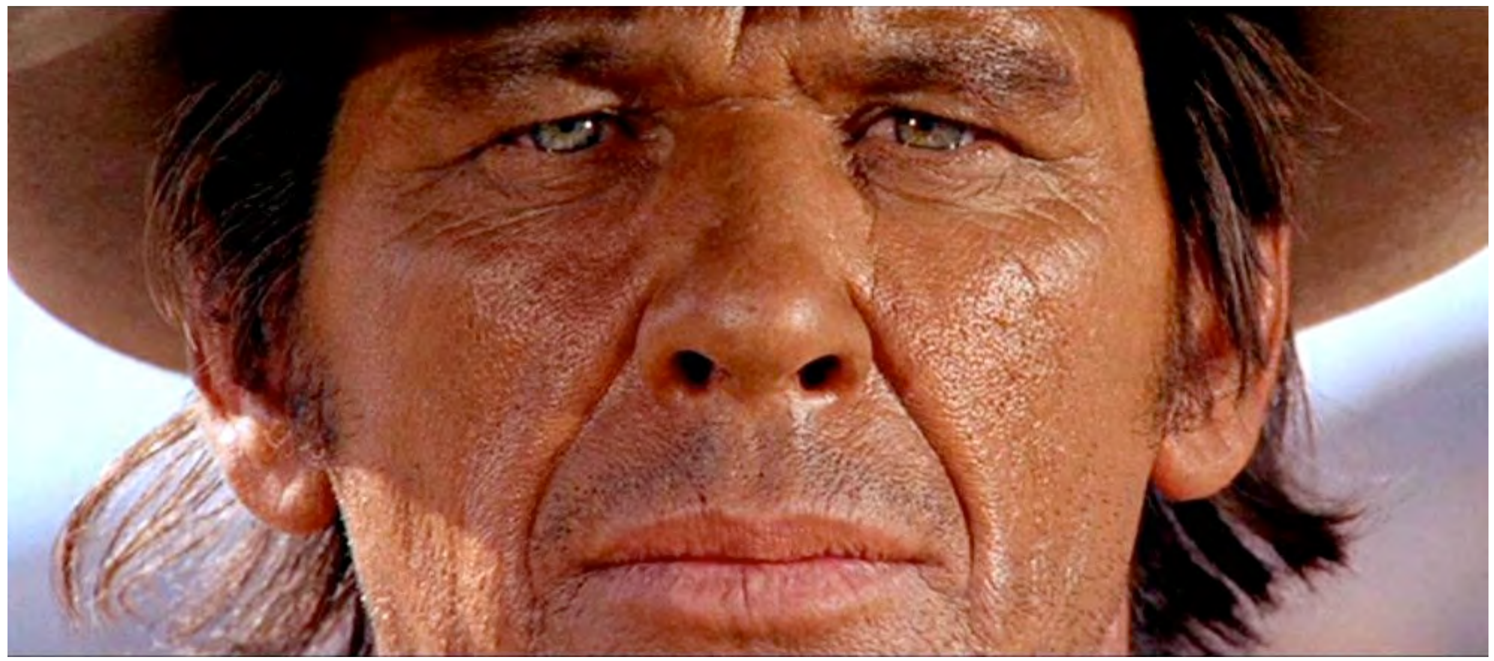

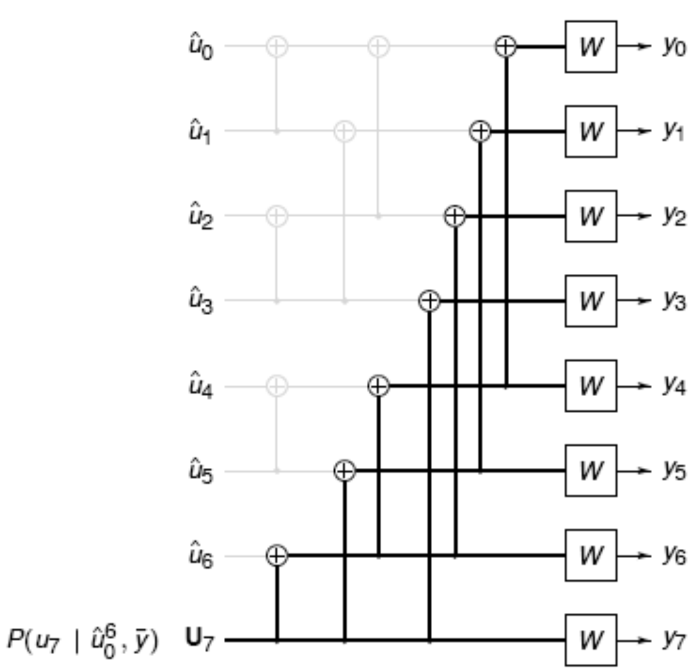
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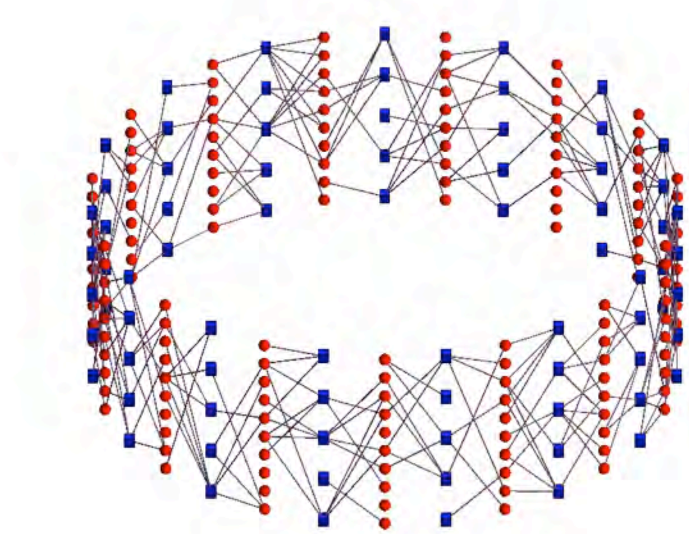
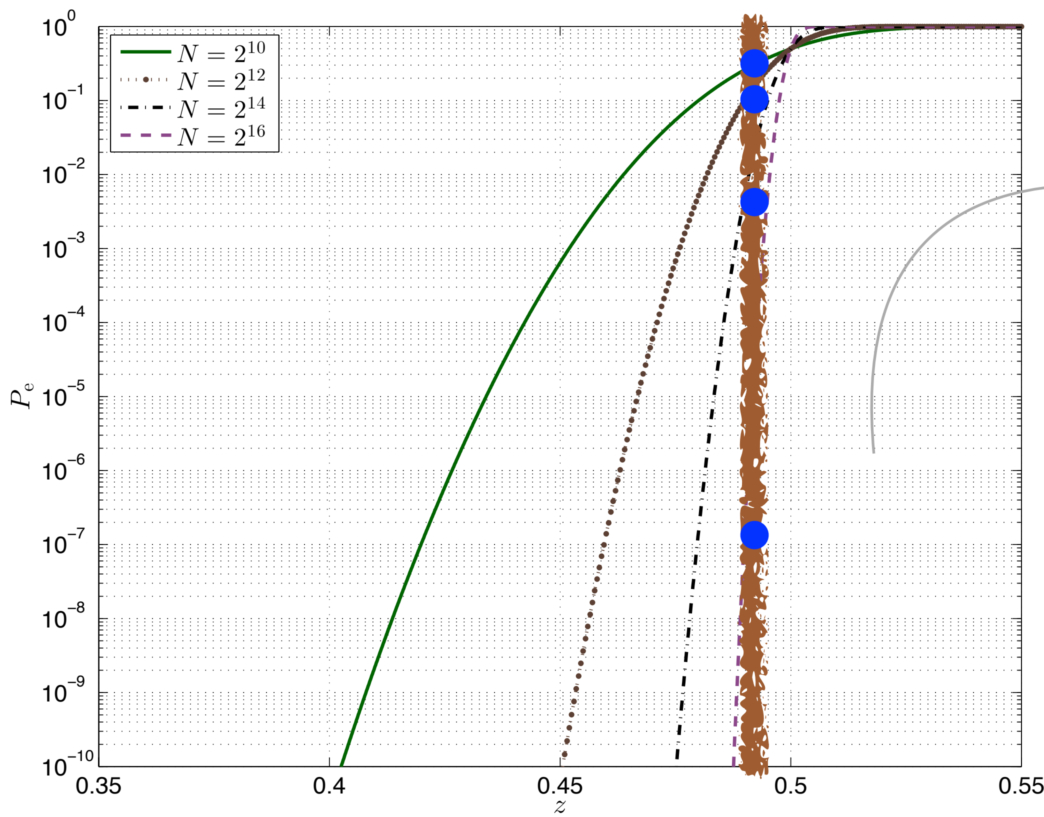


Summary

	optimal	Polar	Spatially Coupled
error exponent	$e^{-NE(R,W)}$	$2^{-\sqrt{N}2^{\frac{\sqrt{\log_2 N}}{2}Q^{-1}(\frac{R}{I(W)})}}$	$e^{-NG(R,W)}$
scaling	N is $\Theta(\frac{1}{(I(W)-R)^2})$	N is $\Theta(\frac{1}{(I(W)-R)^\mu})$ $3.579 \leq \mu \leq 4.714$	N is $\Theta(\frac{1}{(I(W)-R)^3})$
error floor	$P(W) \leq P(W')^{\frac{F(W)}{F(W')}}$	$P(W) \leq P(W')^{\frac{\log Z(W)}{\log Z(W')}}$	$P(W) \leq P(W')^{\frac{G(W)}{G(W')}}$

Error Exponent



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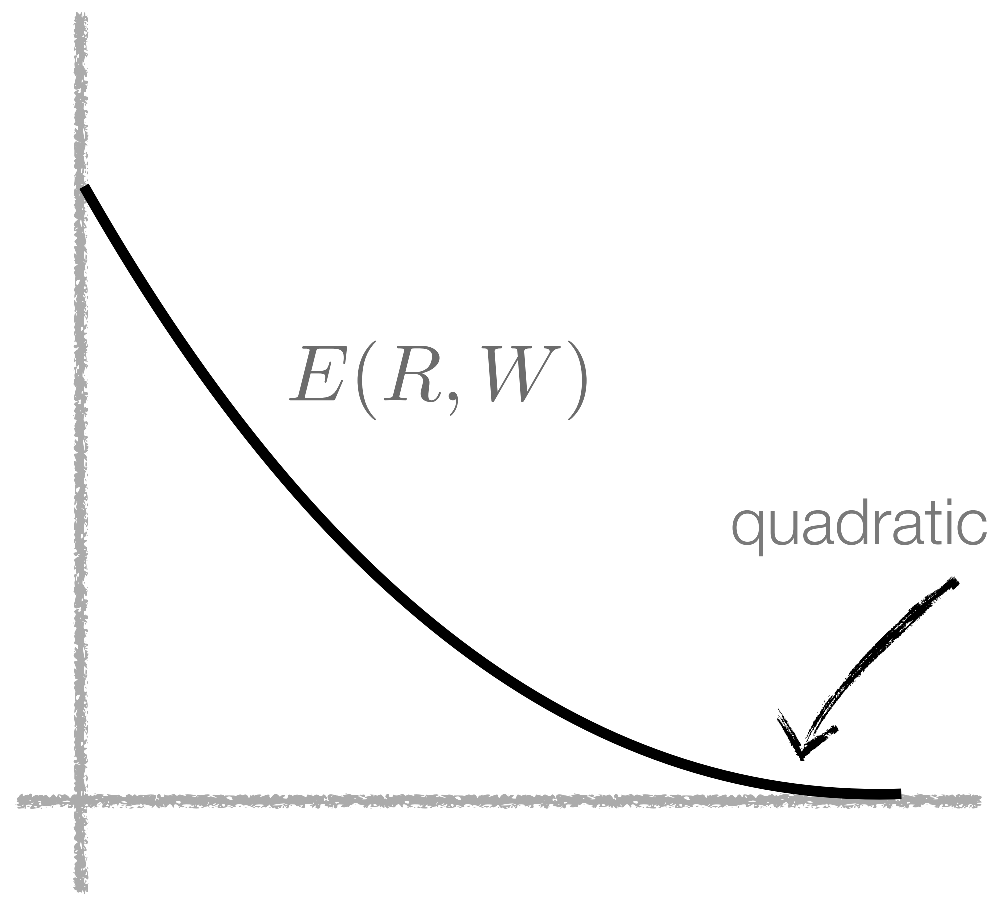

$$P_N(R,W) \sim e^{-NE(R,W)} \qquad E(R,W) = \lim_{N\to\infty} \frac{1}{N} P_N(R,W)$$

[Gallager 1965]

Error Exponent of Polar Codes

$$\lim_{n\to\infty} P\{Z_n \leq 2^{-\sqrt{N}2^{\frac{1}{2}\sqrt{\log_2 N}Q^{-1}(\frac{R}{I(W)})}}\} = R < I(W)$$

[Arikan, Telatar 2009] [Hassani, Mori, Tanaka, U. 2011]

Error Exponent of SC Codes

$$P\{|P_N(G,\epsilon,\ell) - E[P_N(G,\epsilon,\ell)]| \geq \delta\} \leq e^{-\alpha N}$$

G	graph
ϵ	channel parameter
ℓ	# of iterations

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simplest sufficient condition: code has expansion at least 3/4 which is true whp if left degree is at least 5; (less restrictive conditions are known but more complicated);

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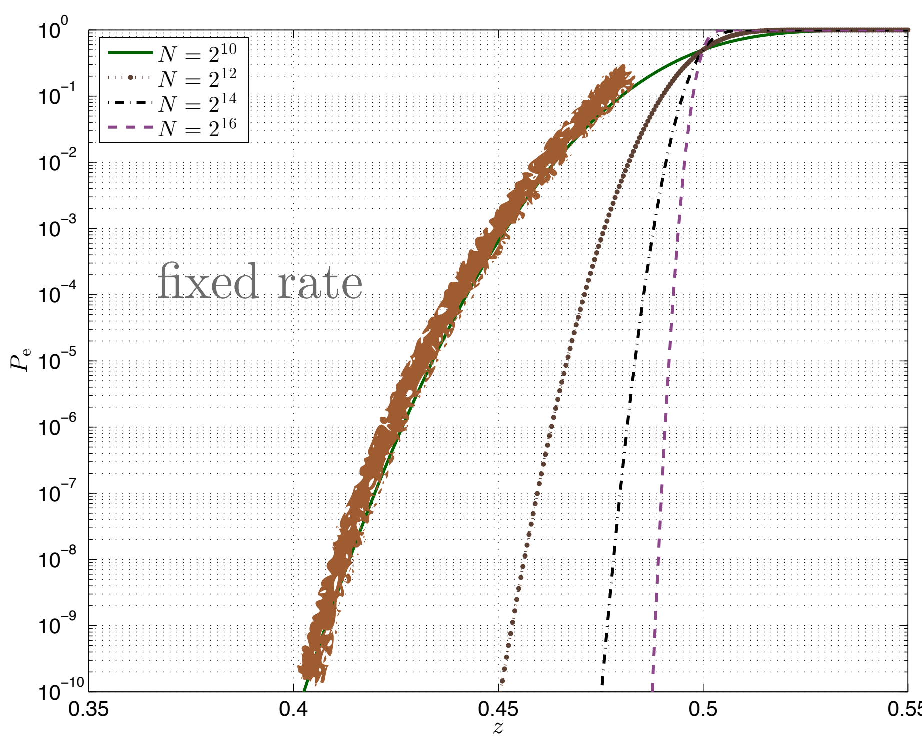

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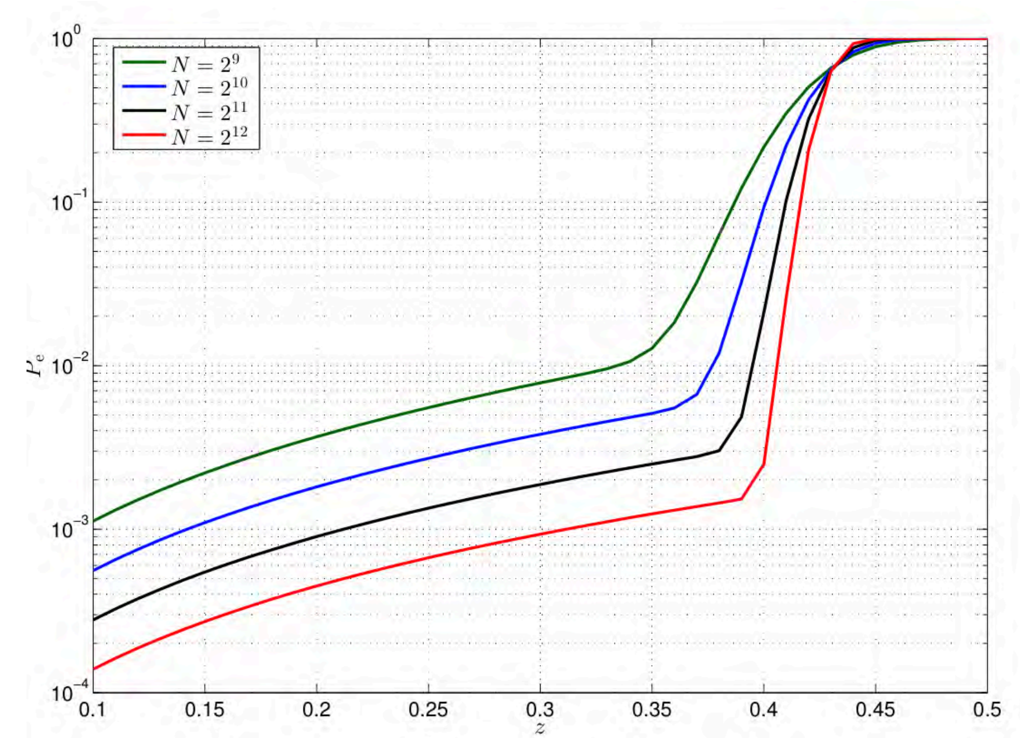

good channel

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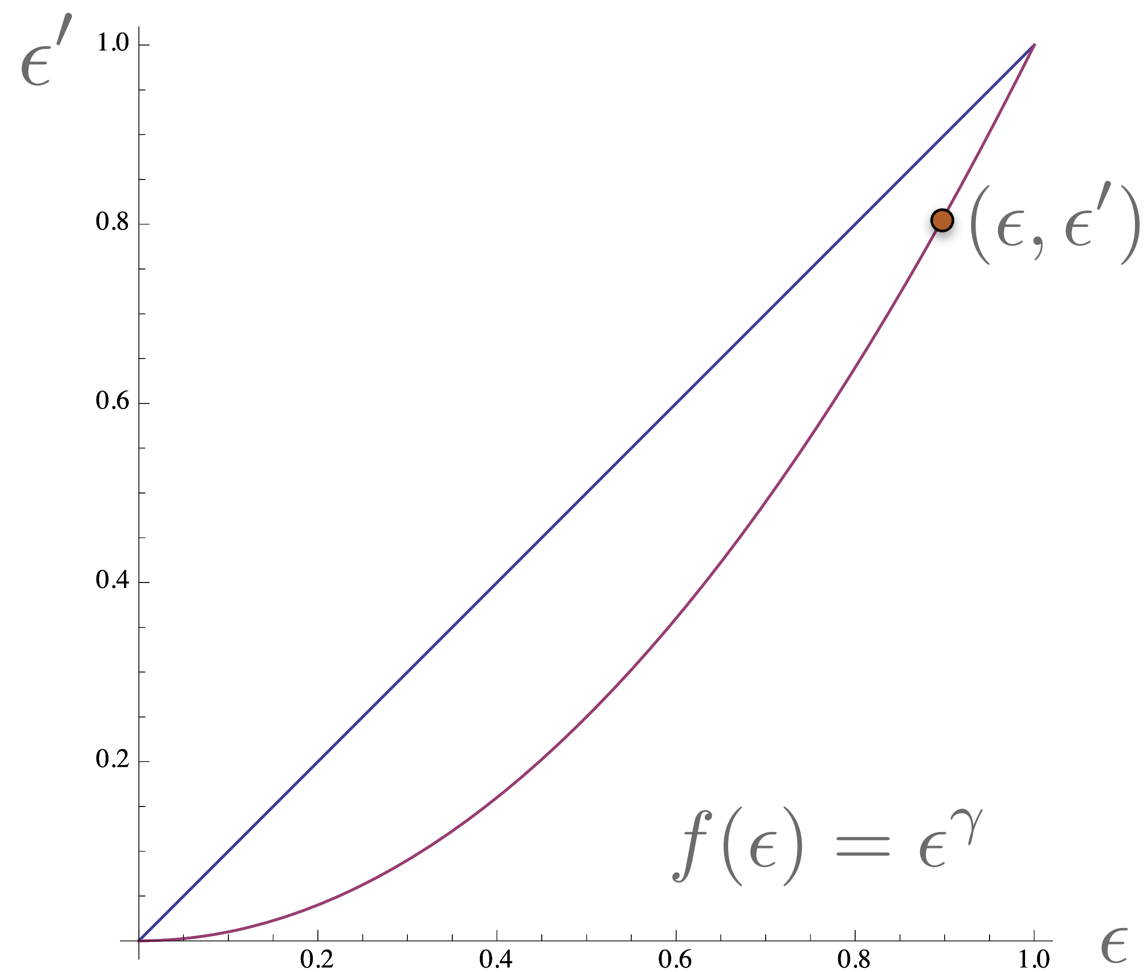
Summary

optimal: $P(W) \leq P(W')^{\frac{F(W)}{F(W')}}$

polar: $P(W) \leq P(W')^{\frac{\log Z(W)}{\log Z(W')}}$ (BMS: if $Z(W) \leq Z(W')^2$)

spatially coupled: $P(W) \leq P(W')^{\frac{G(W)}{G(W')}}$

Error Floor of Polar Codes — BEC



Consider a particular synthetic channel +++-+++ and two starting values ϵ and $\epsilon' \leq \epsilon$.

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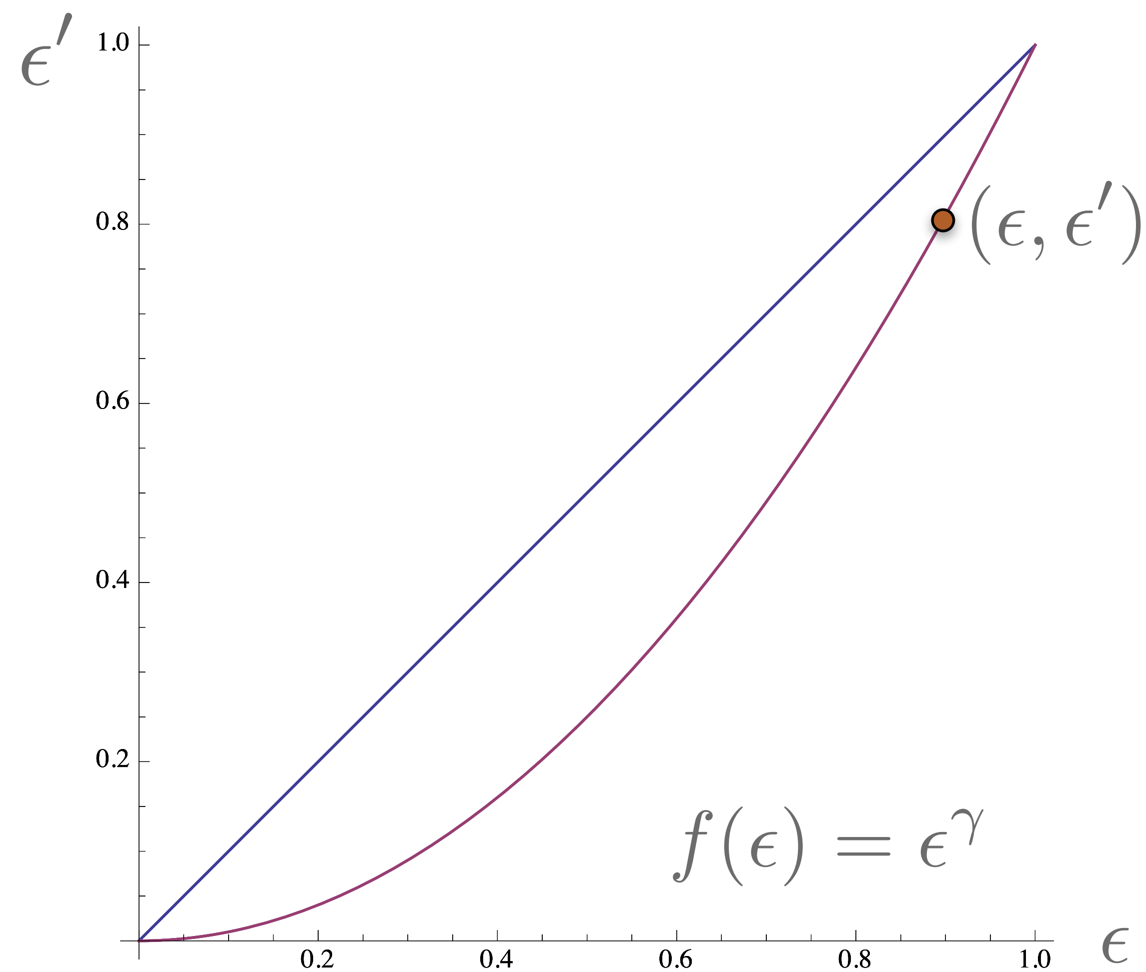

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If $\epsilon' \leq \epsilon^\gamma$, then $z_i(\epsilon') \leq z_i(\epsilon)^\gamma$, $i \geq 0$.

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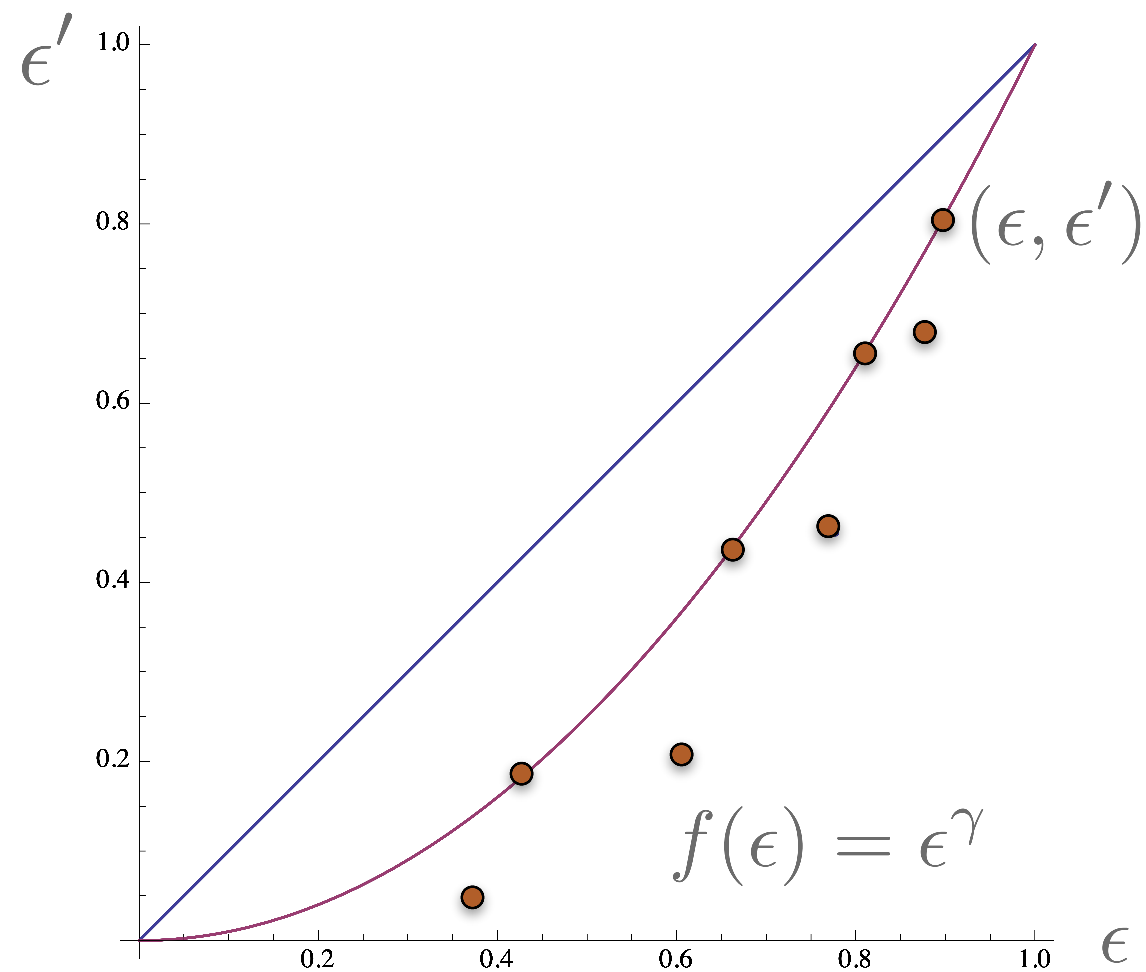

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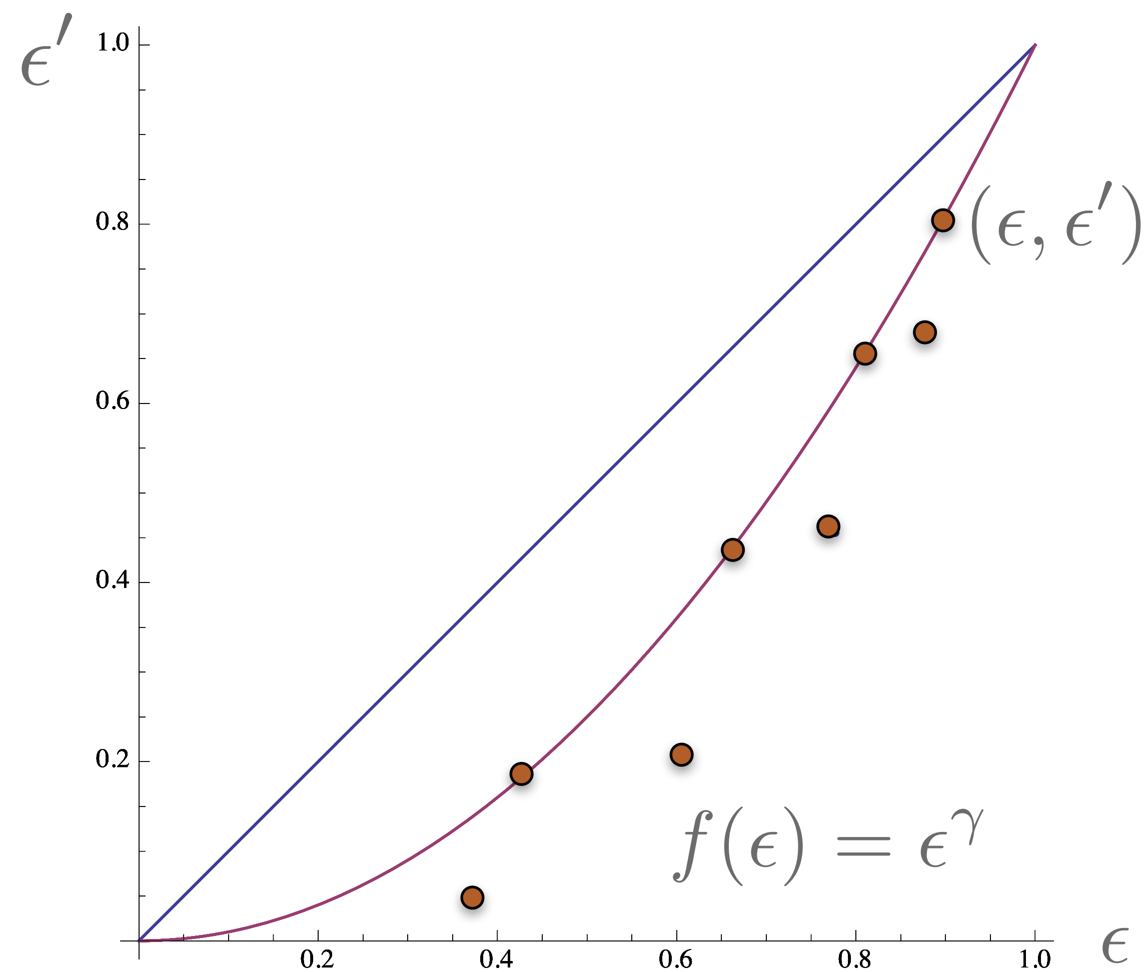

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If $\epsilon' \leq \epsilon^{\gamma}$, then $z_i(\epsilon') \leq z_i(\epsilon)^{\gamma}$, $i \geq 0$.

Hence, if $\tilde{P}(\epsilon) = \sum_{b \in \mathcal{I}} z^{(b)}(\epsilon)$, then

$$\tilde{P}(\epsilon') \leq \tilde{P}(\epsilon)^{\frac{\log(\epsilon')}{\log(\epsilon)}}$$

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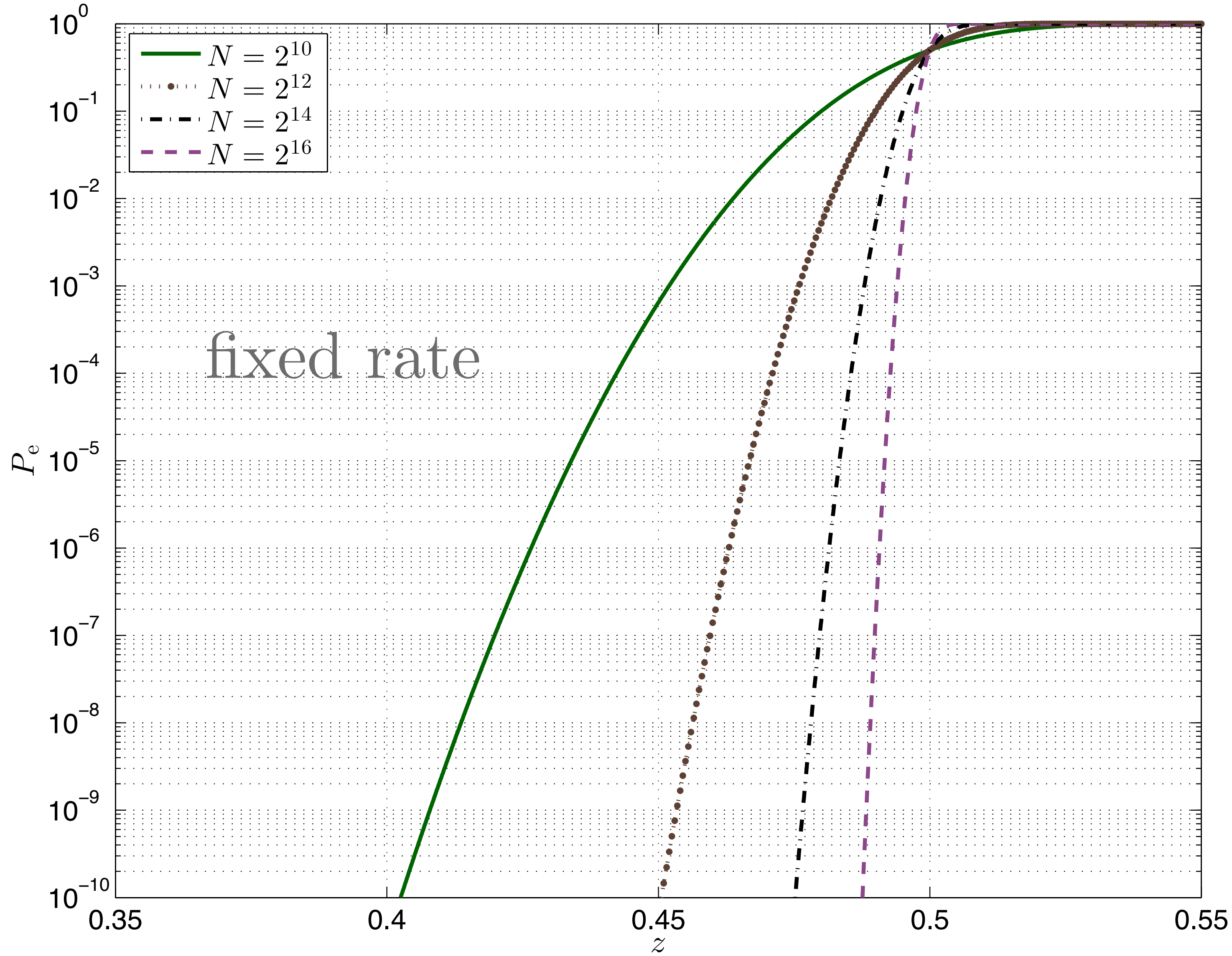
Scaling Exponent

family of codes of increasing N



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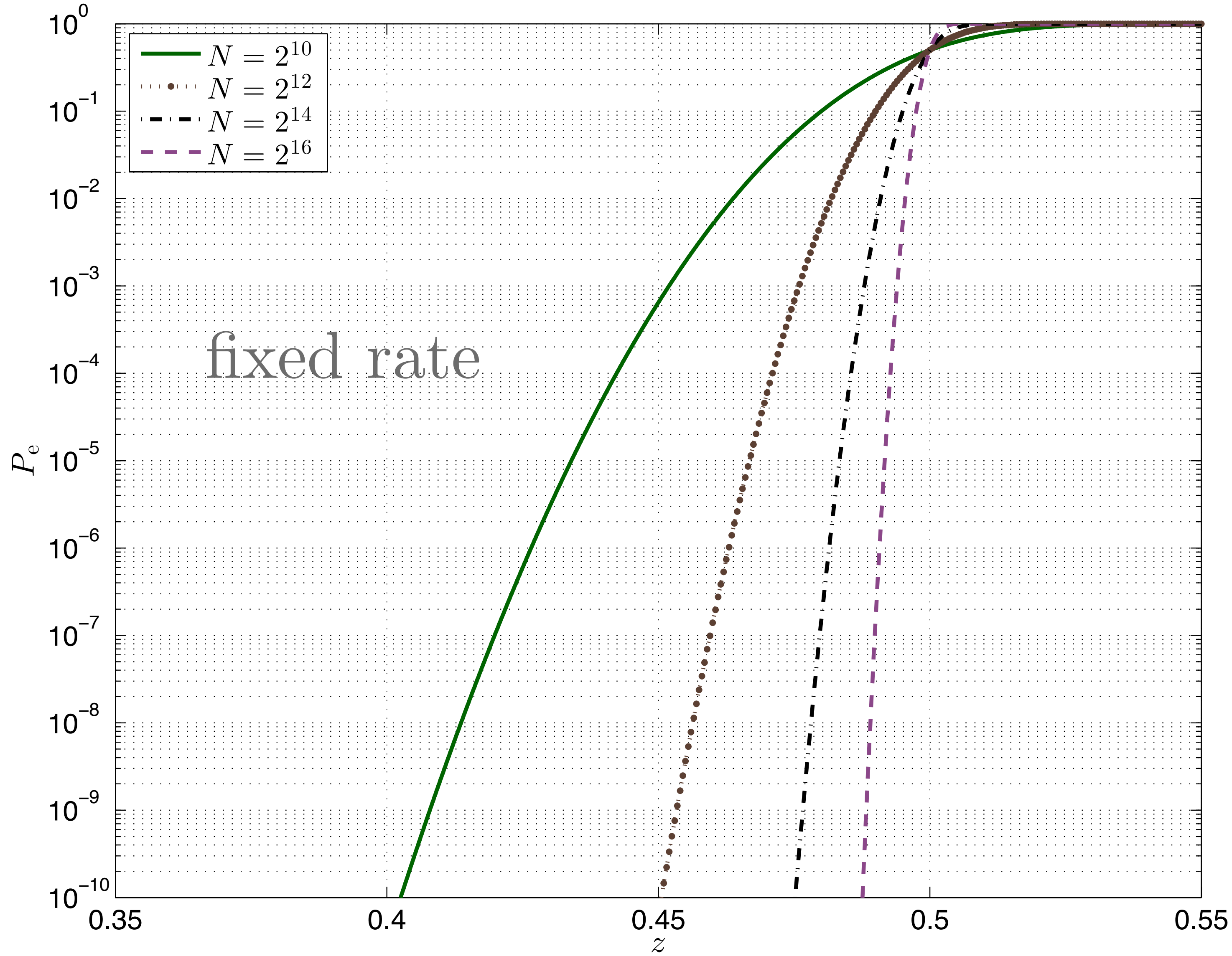

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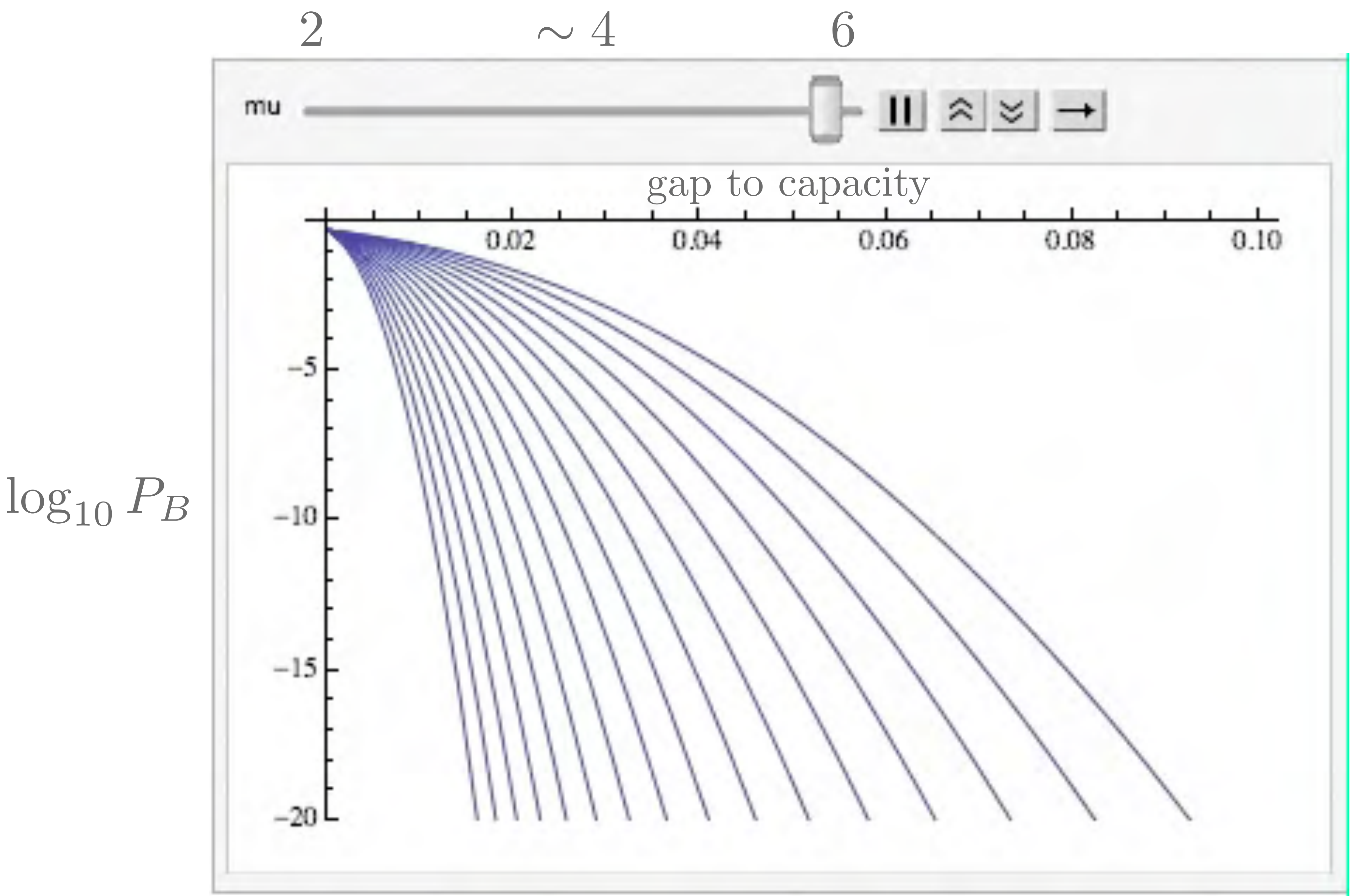

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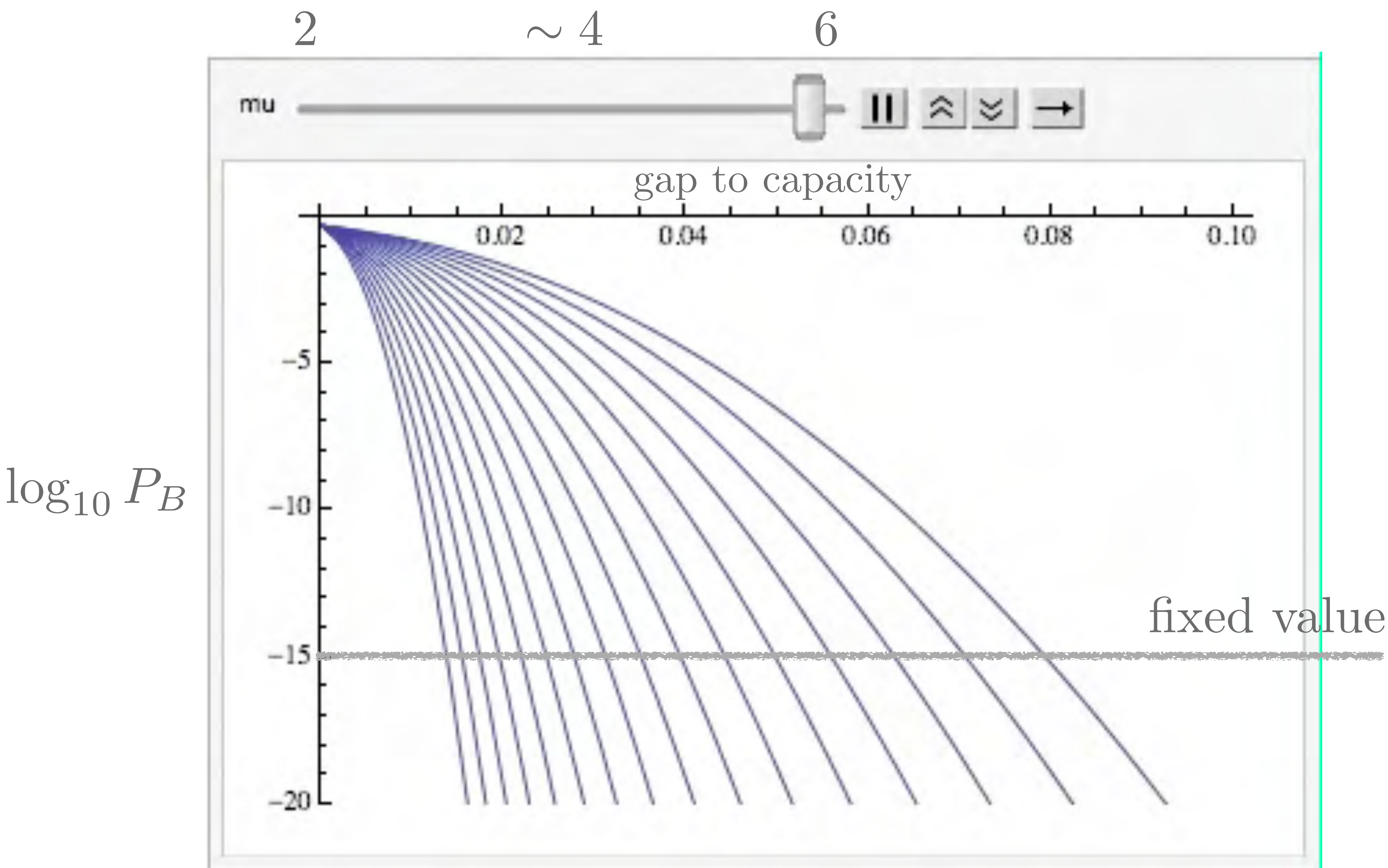

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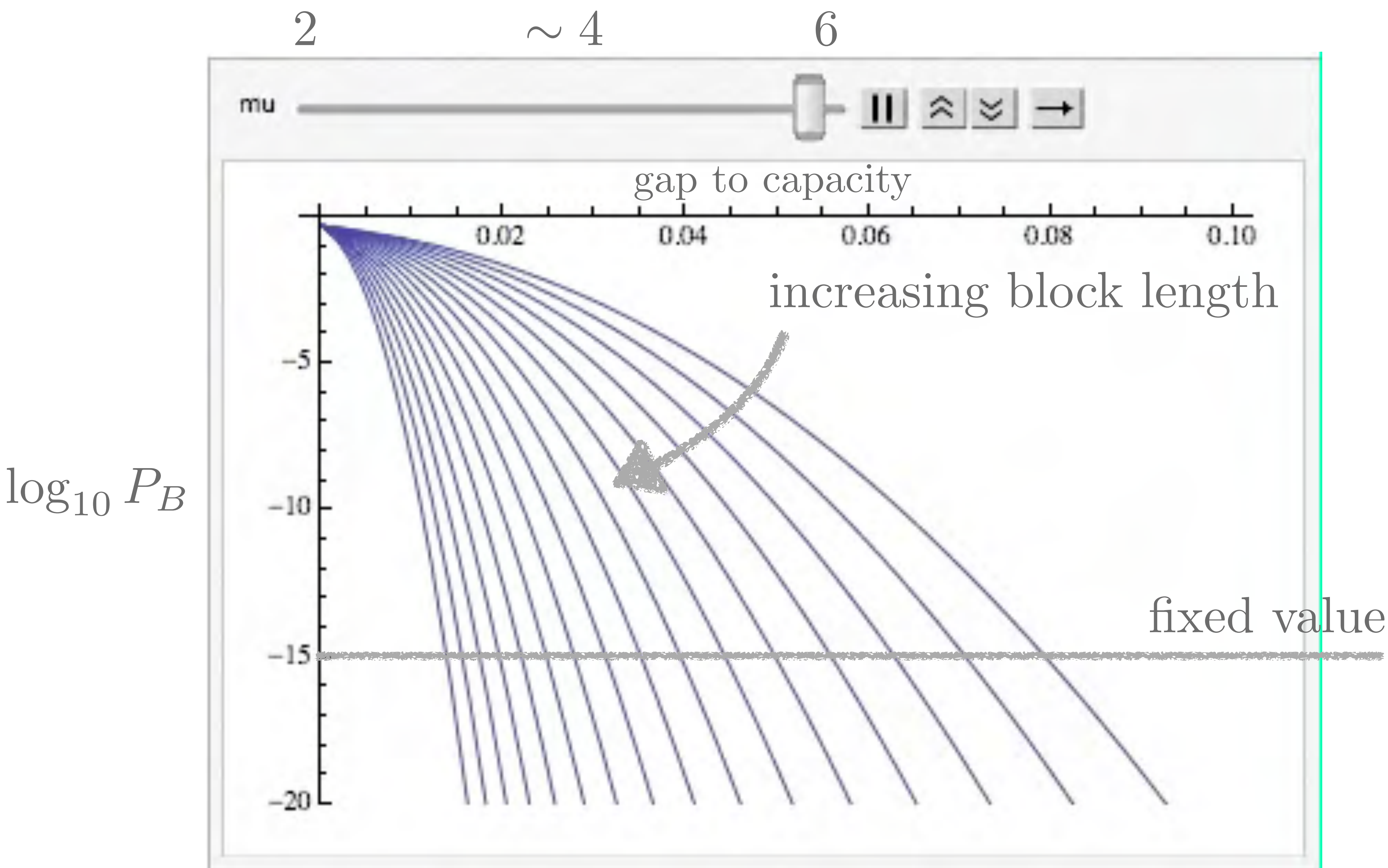

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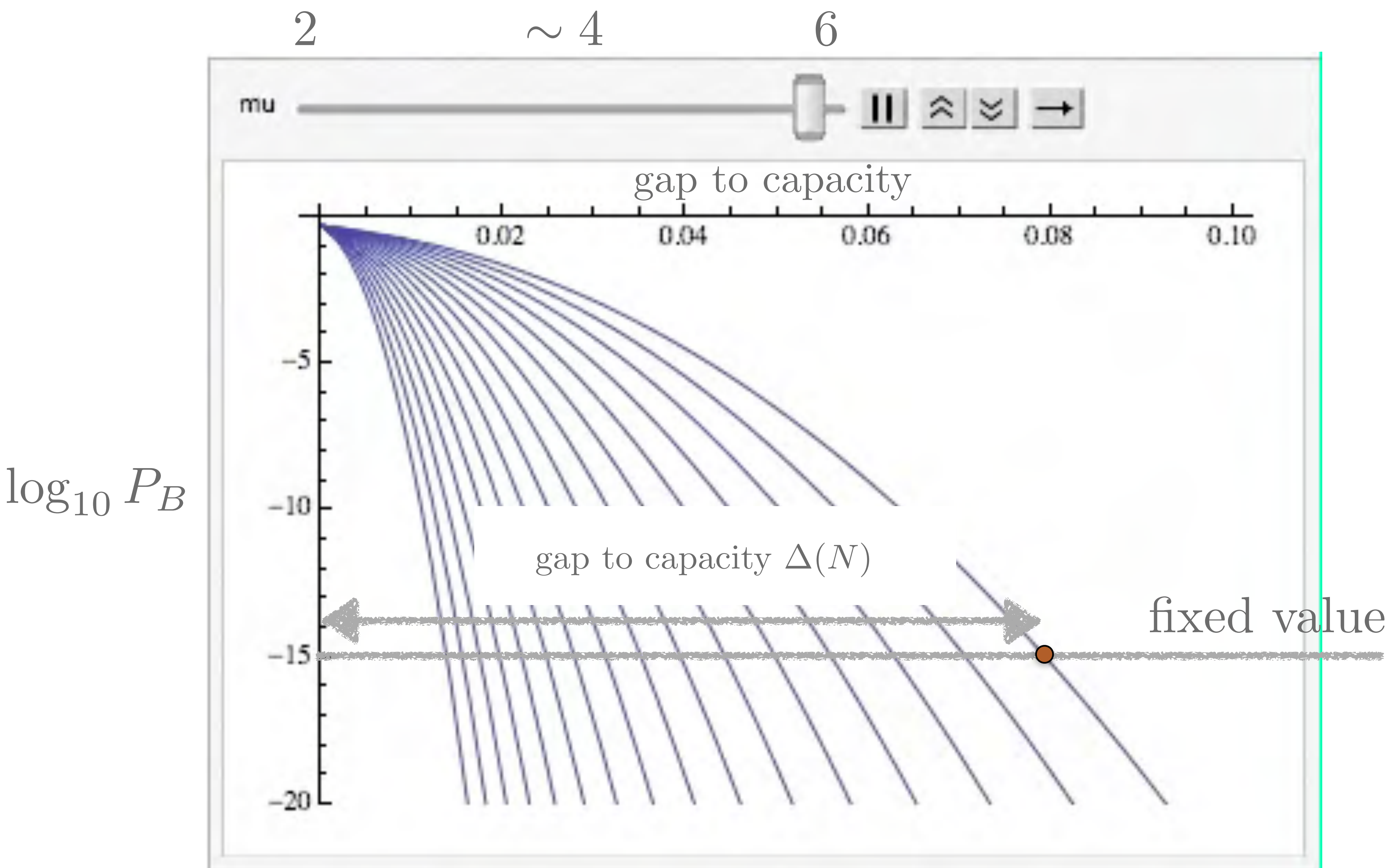
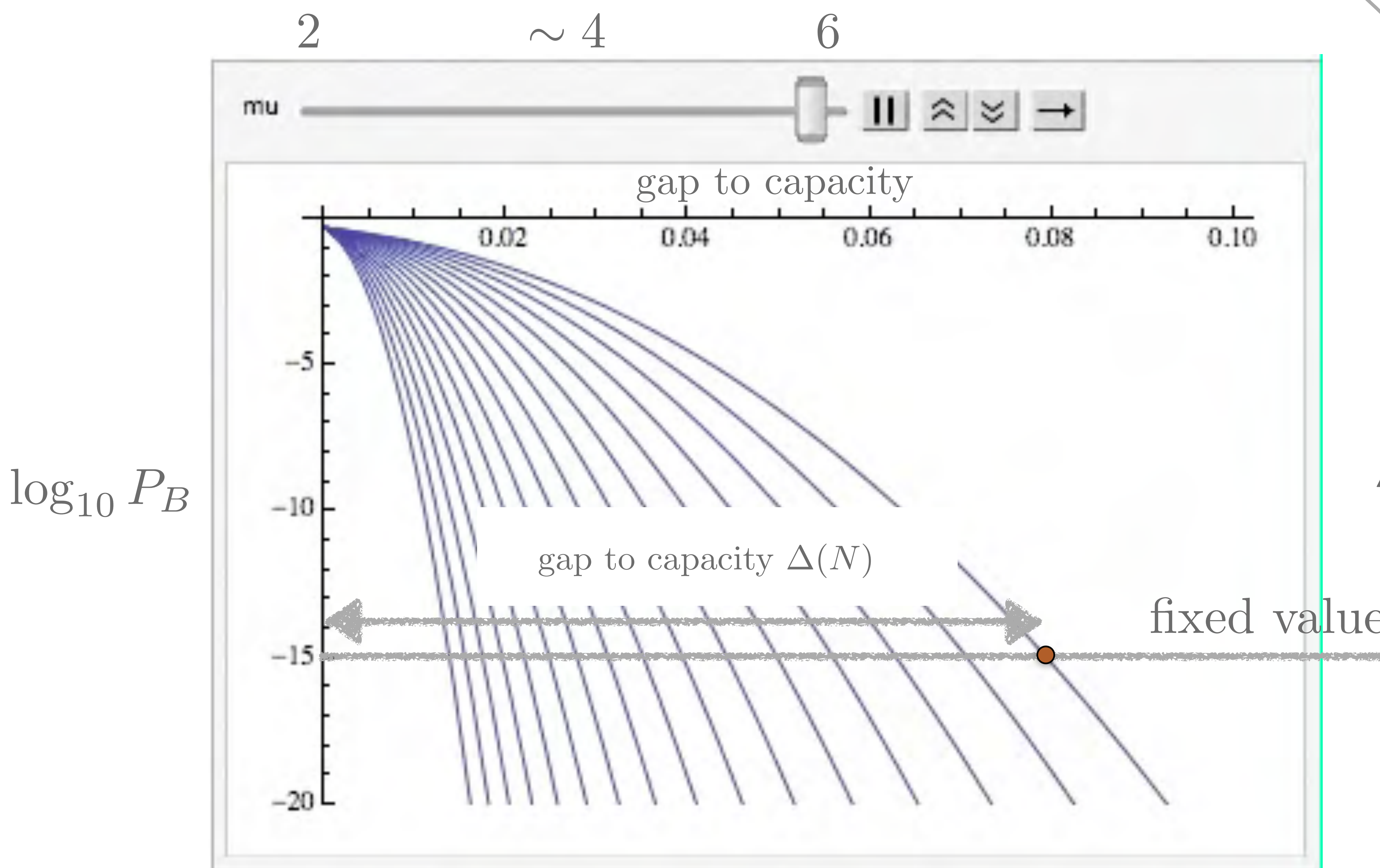


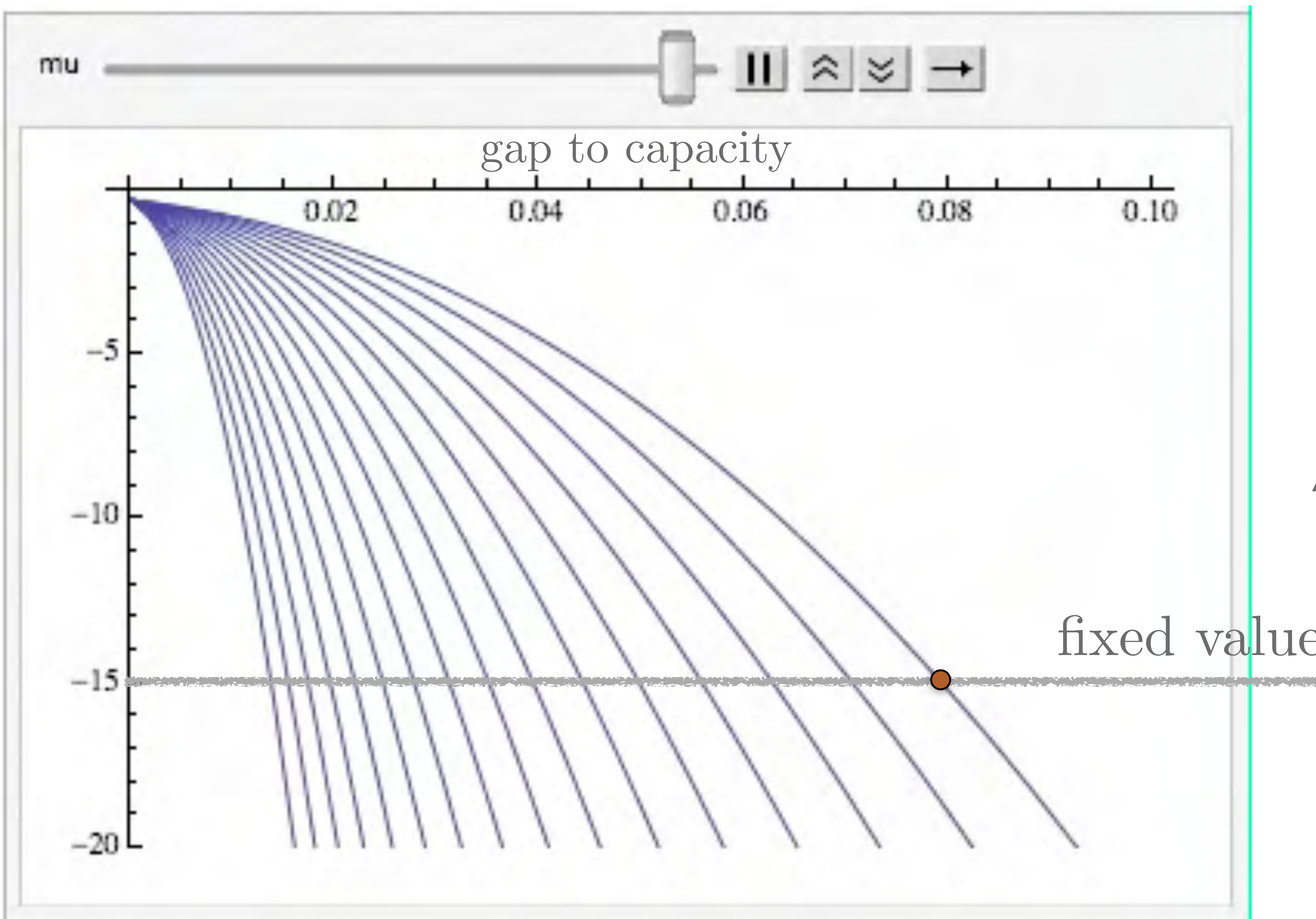
Scaling Exponent μ



$\Delta(N) \sim N^{-\frac{1}{\mu}}$

Scaling Exponent μ

2 ~ 4 6

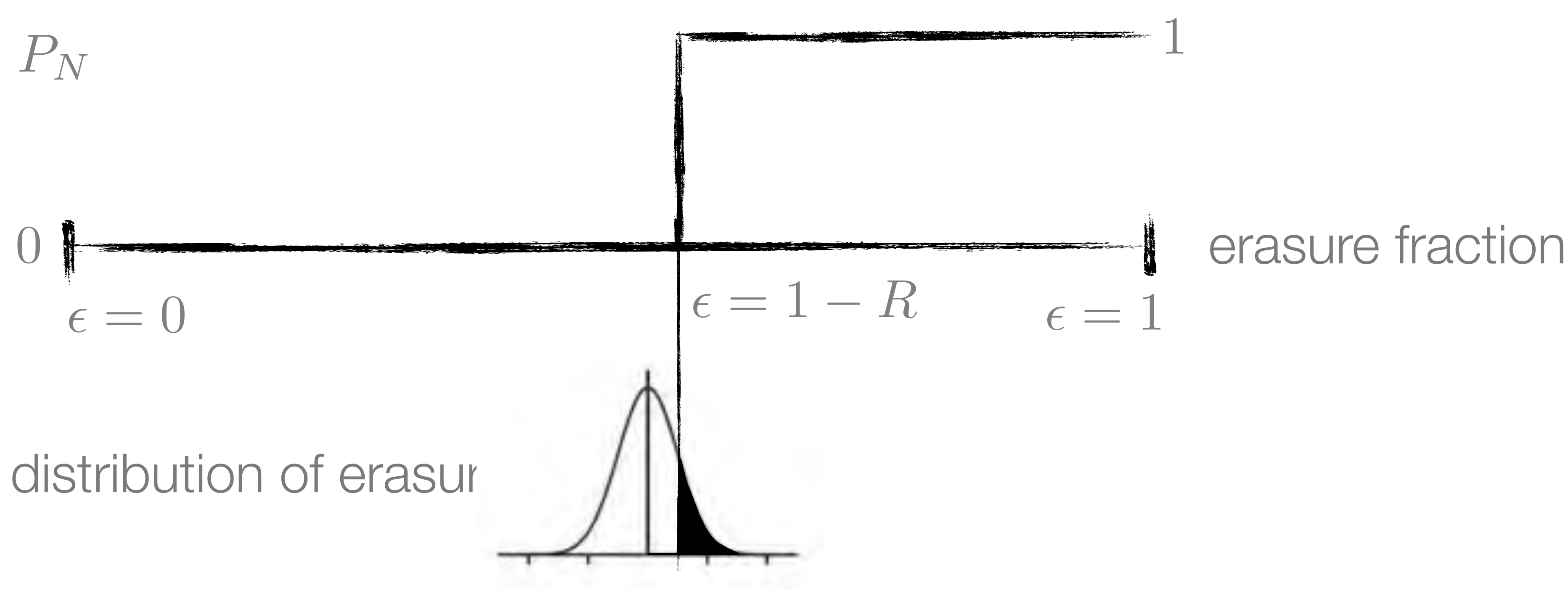


$\log_{10} P_B$

$\Delta(N) \sim N^{-\frac{1}{\mu}}$

Gedankenexperiment -- BEC

``perfect code"



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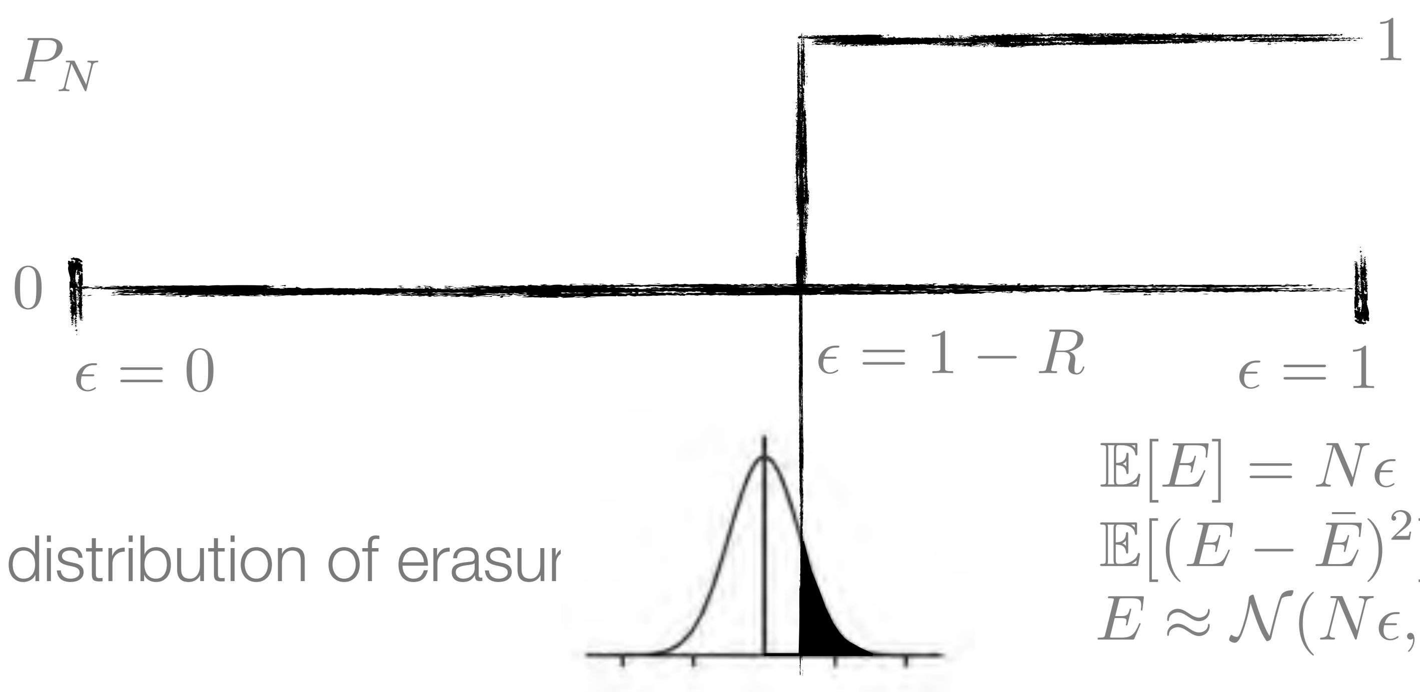

$$\mathbb{E}[E] = N\epsilon$$
$$\mathbb{E}[(E - \bar{E})^2] = N\epsilon(1-\epsilon)$$
$$E \approx \mathcal{N}(N\epsilon, N\epsilon(1-\epsilon))$$

Gedankenexperiment -- BEC

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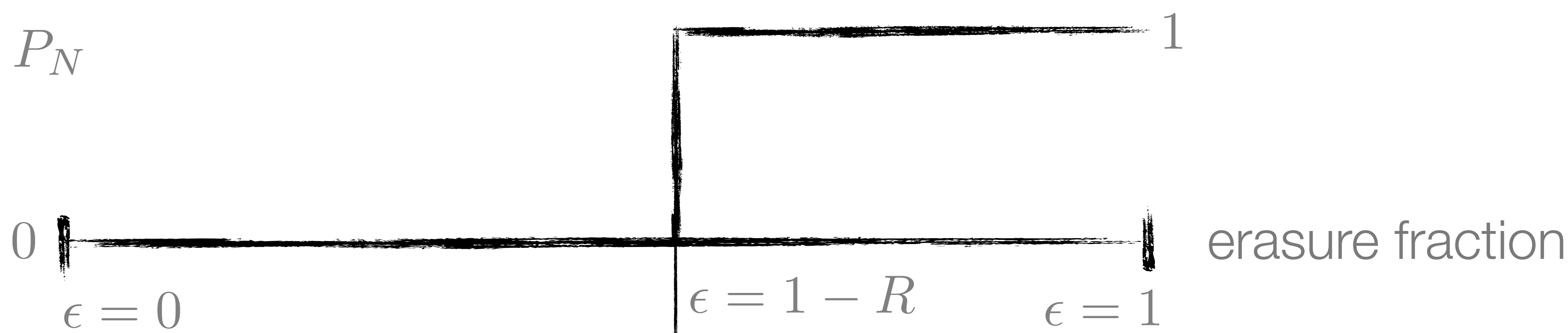
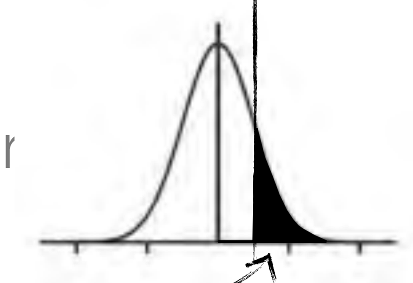


distribution of erasur



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$$P_N \approx Q\Big(\frac{N((1-R)-\epsilon)}{\sqrt{N\epsilon(1-\epsilon)}}\Big) = Q\Big(\frac{\sqrt{N}((1-R)-\epsilon)}{\sqrt{\epsilon(1-\epsilon)}}\Big) = Q\Big(\frac{z}{\sqrt{\epsilon(1-\epsilon)}}\Big)$$

$$z = \sqrt{N}((1-R)-\epsilon)$$

Scaling of Block Codes under MAP -- BEC

random linear block codes are almost perfect

00101010001
01110010010
10101010101
01000101001
01011100010

square binary
random matrix
of dimension n

Scaling of Block Codes under MAP -- BEC

random linear block codes are almost perfect

$$\boxed{\begin{matrix}00101010001\\01110010010\\10101010101\\01000101001\\01011100010\end{matrix}}$$

square binary random matrix of dimension n

probability that full rank $$\prod_{i=0}^{n-1} \frac{2^n - 2^i}{2^n} = \prod_{i=0}^{n-1} (1 - 2^{i-n}) \overset{n\to\infty}{\to} 0.28878809508\ldots$$

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$$\boxed{\begin{matrix}00101010001\\01110010010\\10101010101\\01000101001\end{matrix}}$$

if we have k rows
less then probability
decays by roughly

$$2^{-\binom{k+1}{2}}$$

hence for random linear block codes the transition is of constant (on an absolute scale) width

Scaling of Optimal Codes

$$N \text{ is } \Theta\left(\frac{1}{(I(W)-R)^2}\right)$$

[Strassen 1962] [Polyanskiy, Poor, Verdú 2009]

Scaling Exponent for Polar Codes

$$N \text{ is } \Theta\left(\frac{1}{(I(W)-R)^{\mu}}\right)$$

$$3.579 \leq \mu \leq 6 \text{ for any BMS channel}$$

[Hassani, Alishahi, U. 2013]

$$\mu \leq 5.702 \text{ for any BMS channel}$$

[Goldin, Burshtein 2013]

BEC:

Assume $h(x)$ is s.t. $h(0) = h(1) = 0$, $h(x) > 0$ for $x \in (0,1)$, and

$$\sup_{x\in(0,1)} \frac{h(x^2)+h(2x-x^2)}{2h(x)} < 2^{-1/\mu^*}$$

Then, $\mu \leq \mu^*$. The value $\mu^* = 3.635$ is achievable.

BMSC:

Assume $h(x)$ is s.t. $h(0) = h(1) = 0$, $h(x) > 0$ for $x \in (0,1)$, and

$$\sup_{x\in(0,1),\, y\in[x\sqrt{2-x^2},\, 2x-x^2]} \frac{h(x^2)+h(y)}{2h(x)} < 2^{-1/\mu^*}$$

Then, $\mu \leq \mu^*$. The value $\mu^* = 4.71$ is achievable.

[Hassani, Mondelli, U. 2014]

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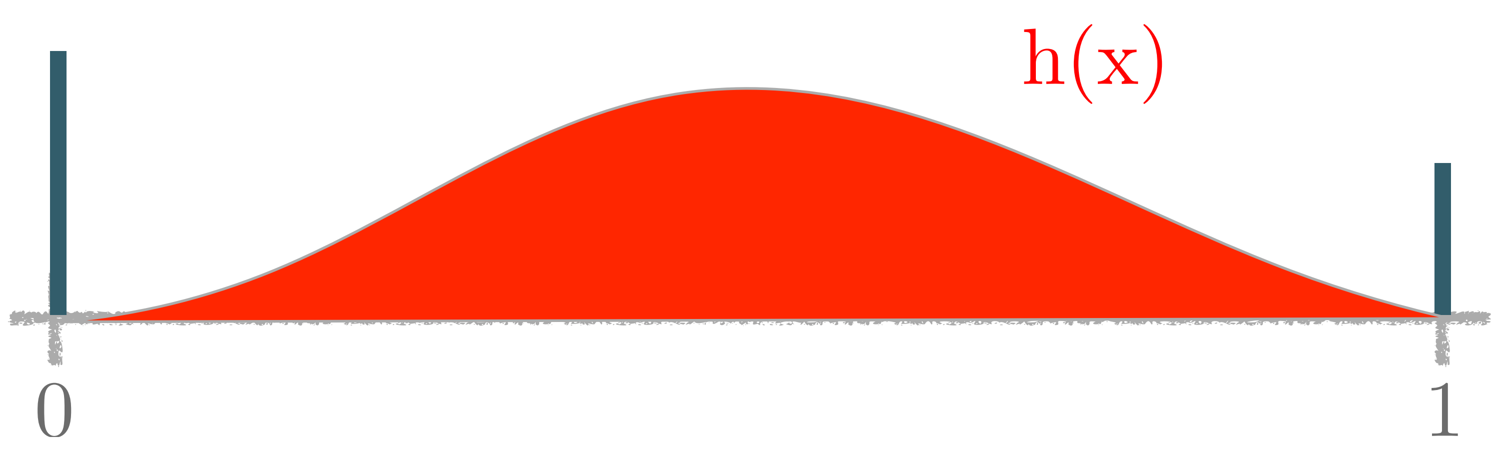

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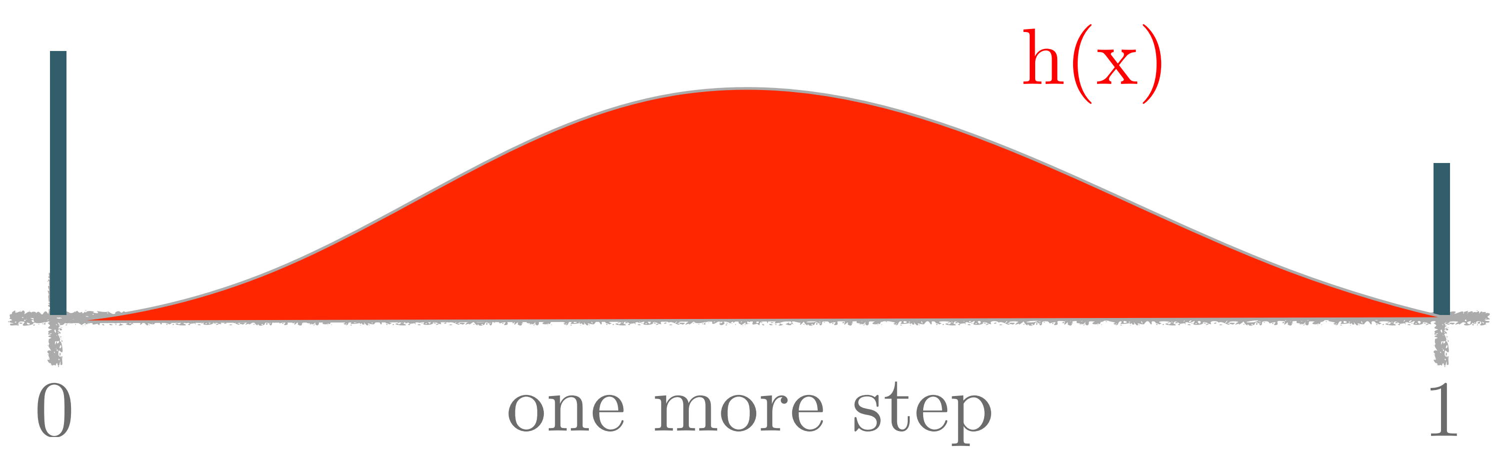

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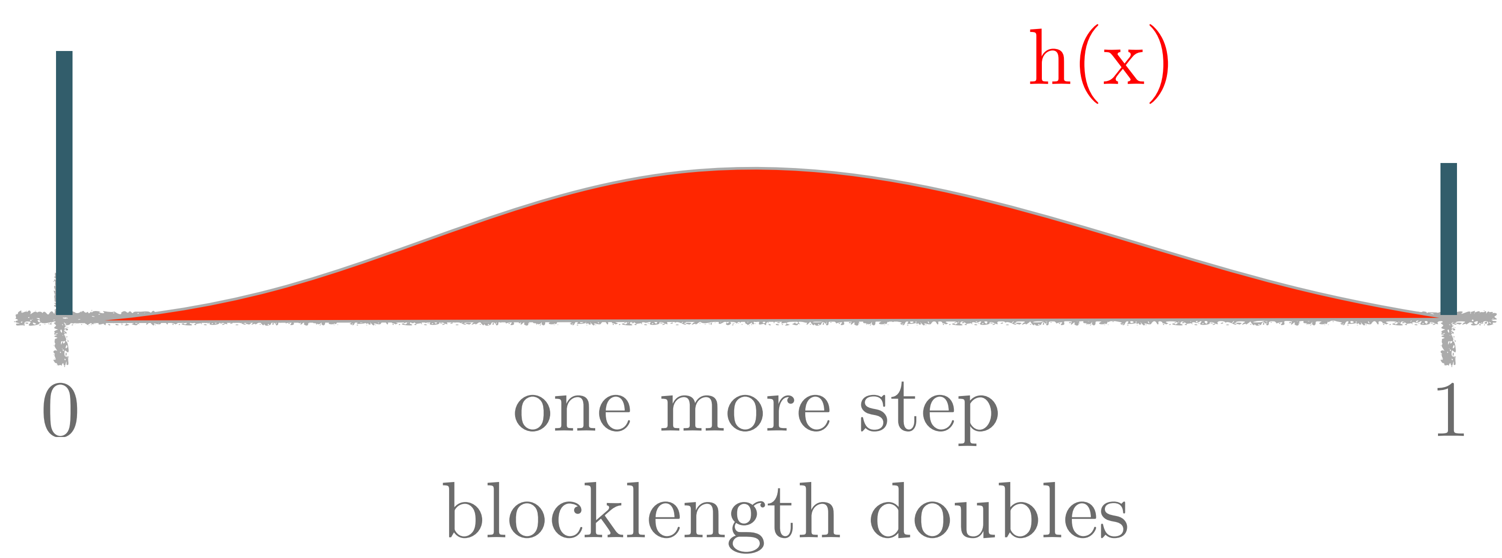
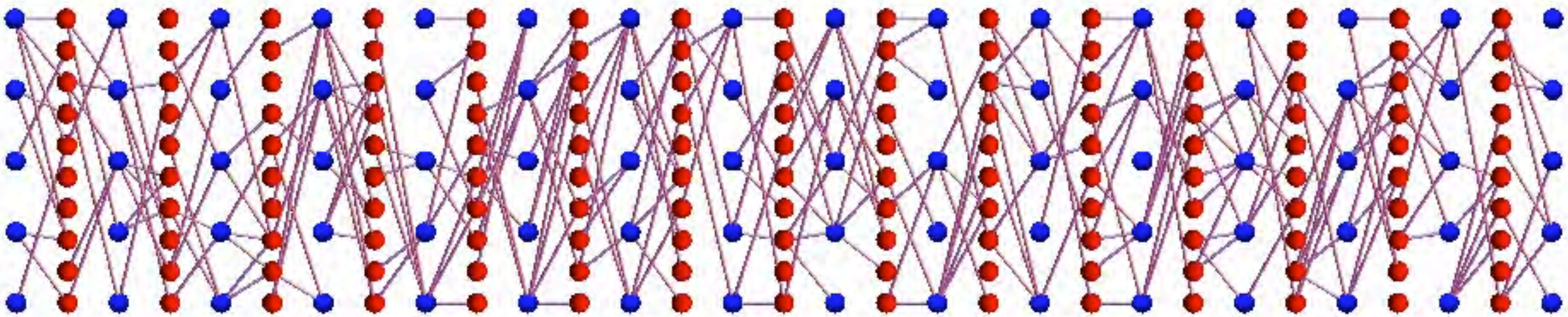
$$\sup_{x \in (0,1)} \frac{h(x^2) + h(2x - x^2)}{2h(x)} < 2^{-1/\mu^*}$$

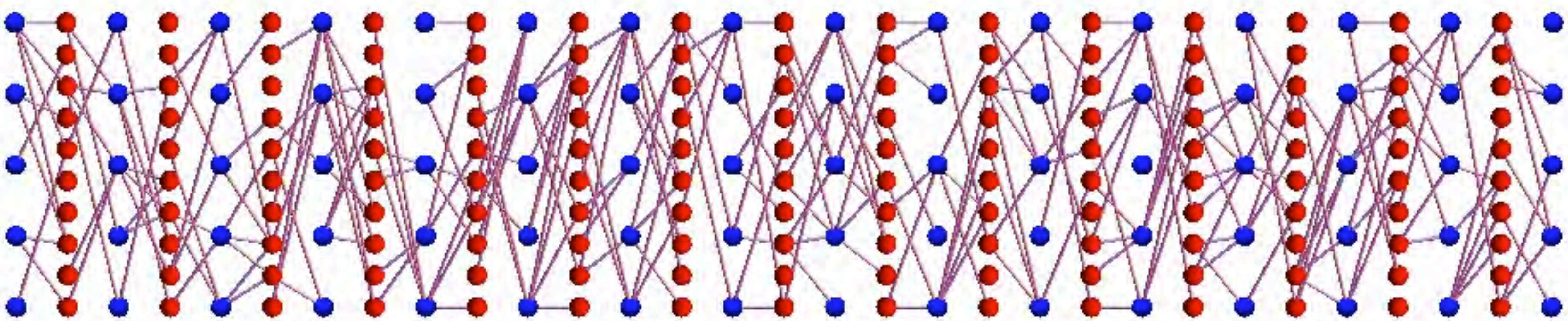
Then, $\mu \leq \mu^*$. The value $\mu^* = 3.635$ is achievable.



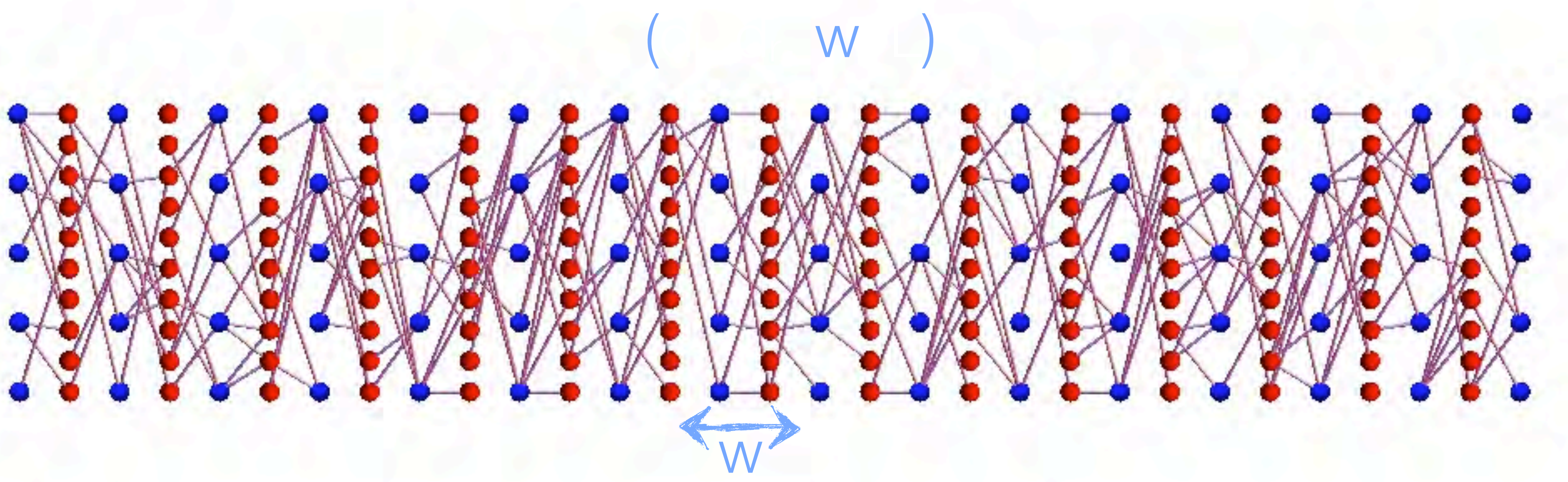
Scaling Exponent for Spatially Coupled Codes



Scaling Exponent for Spatially Coupled Codes

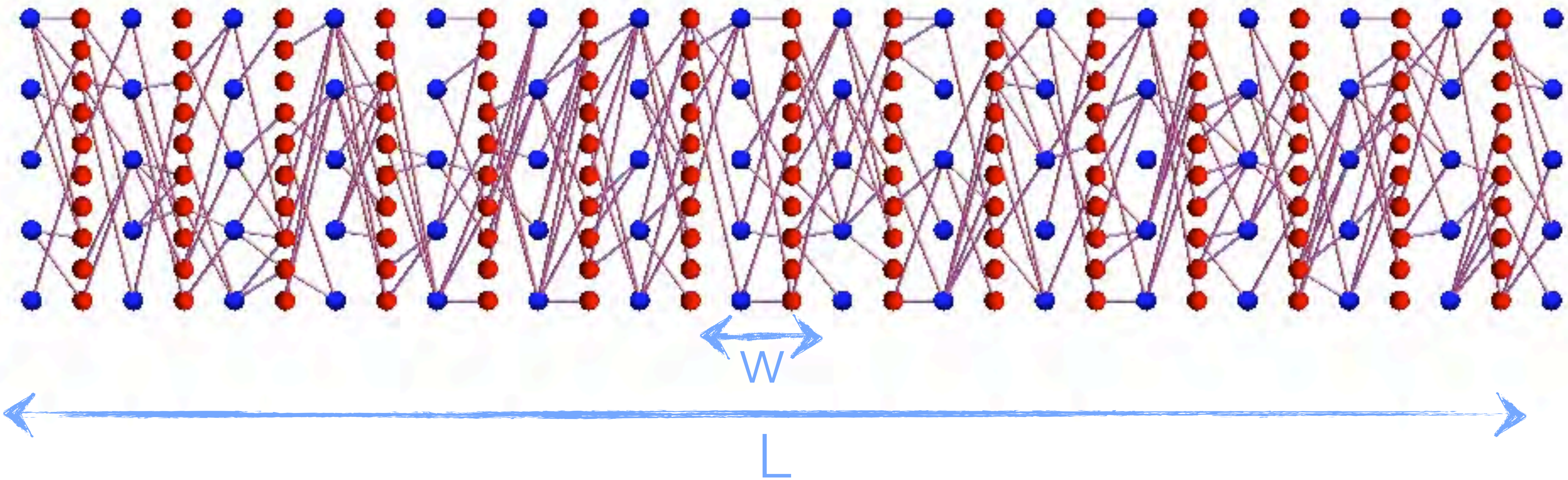


Scaling Exponent for Spatially Coupled Codes

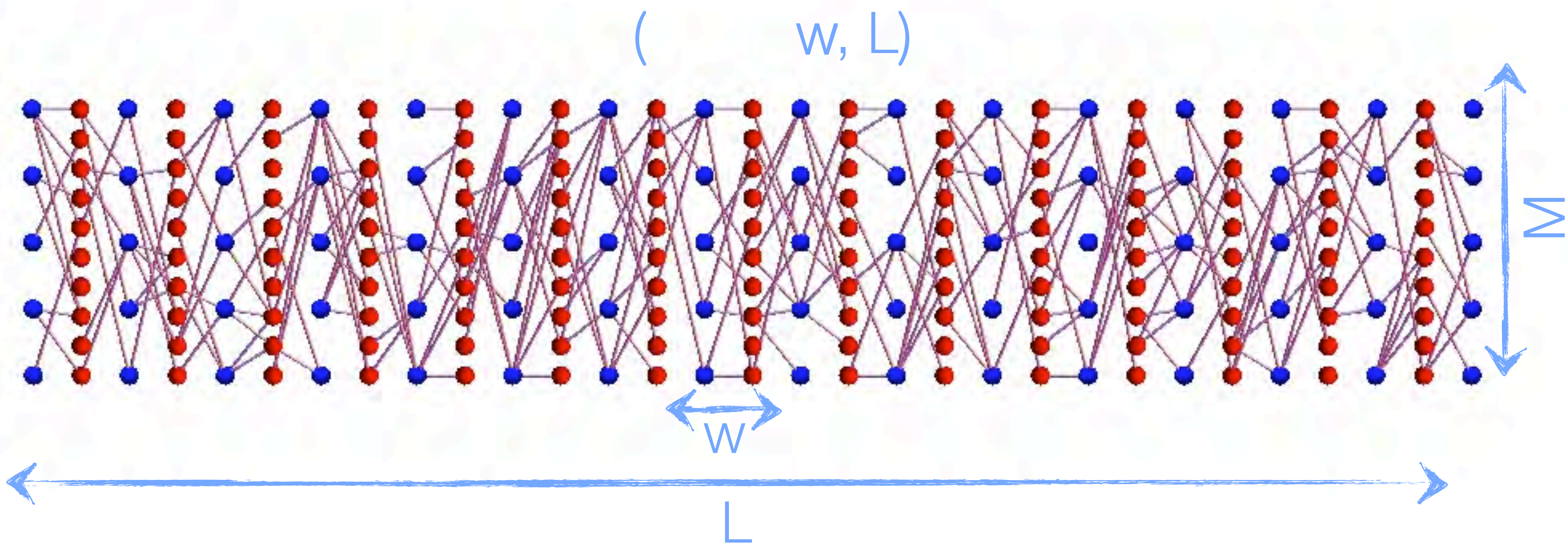


Scaling Exponent for Spatially Coupled Codes

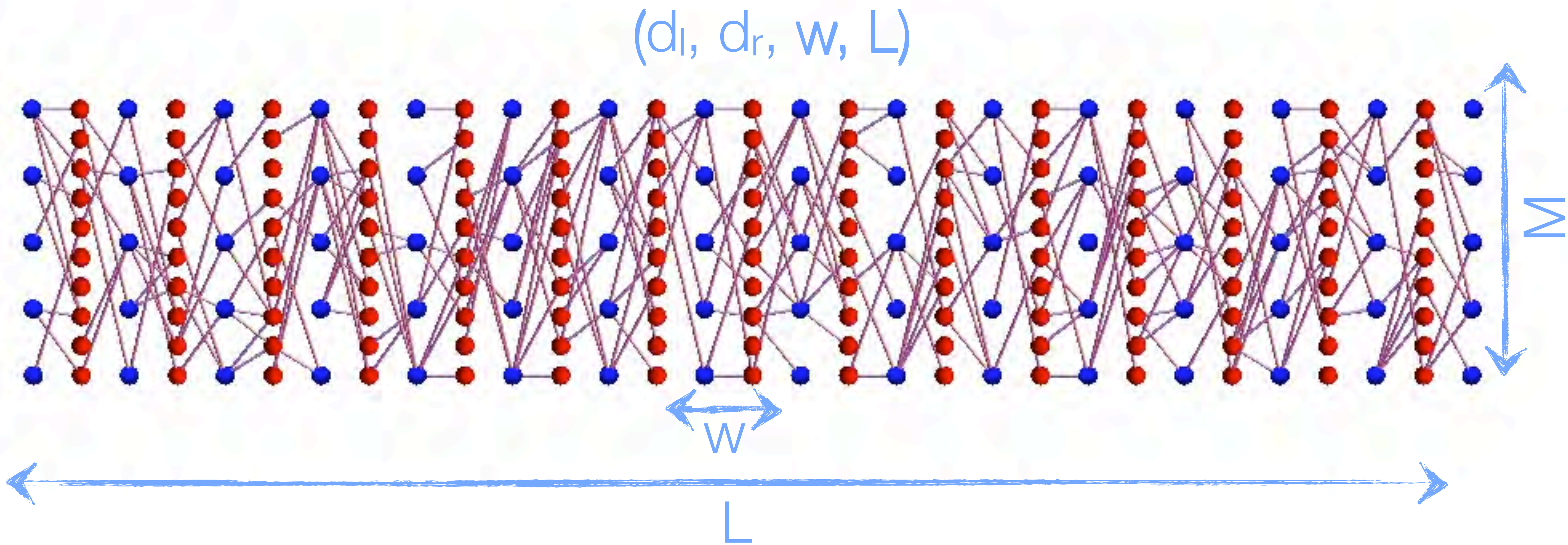
(w, L)



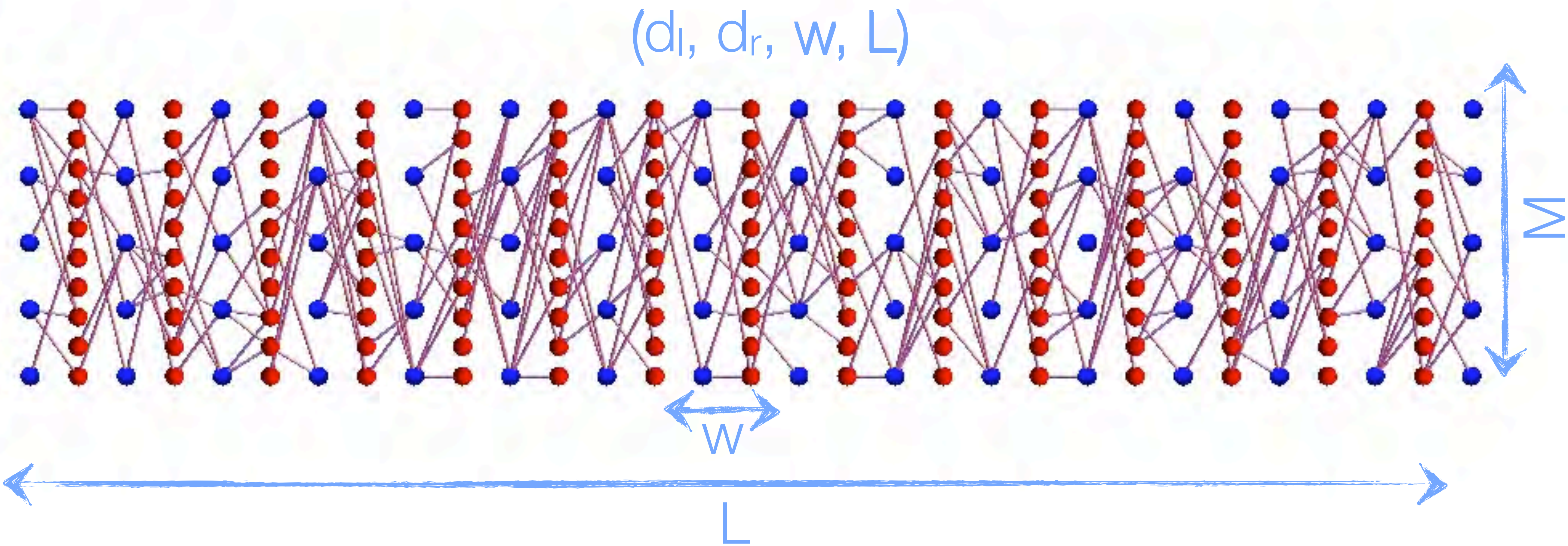
Scaling Exponent for Spatially Coupled Codes



Scaling Exponent for Spatially Coupled Codes

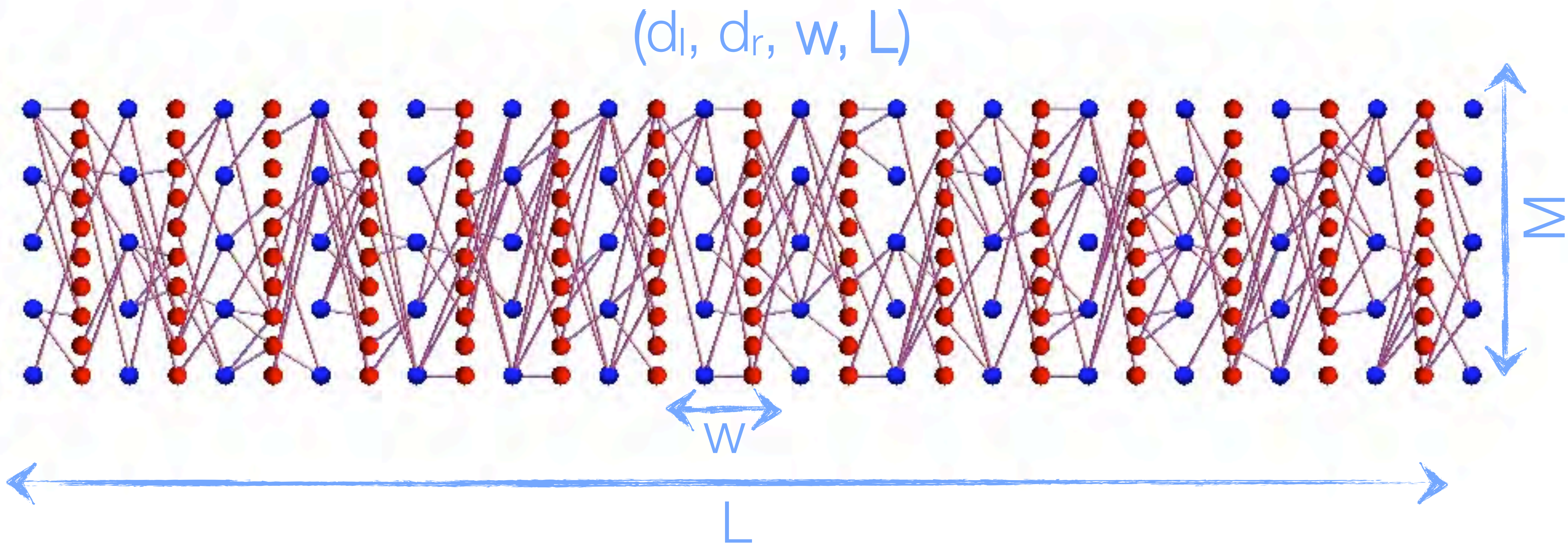


Scaling Exponent for Spatially Coupled Codes



N=L M

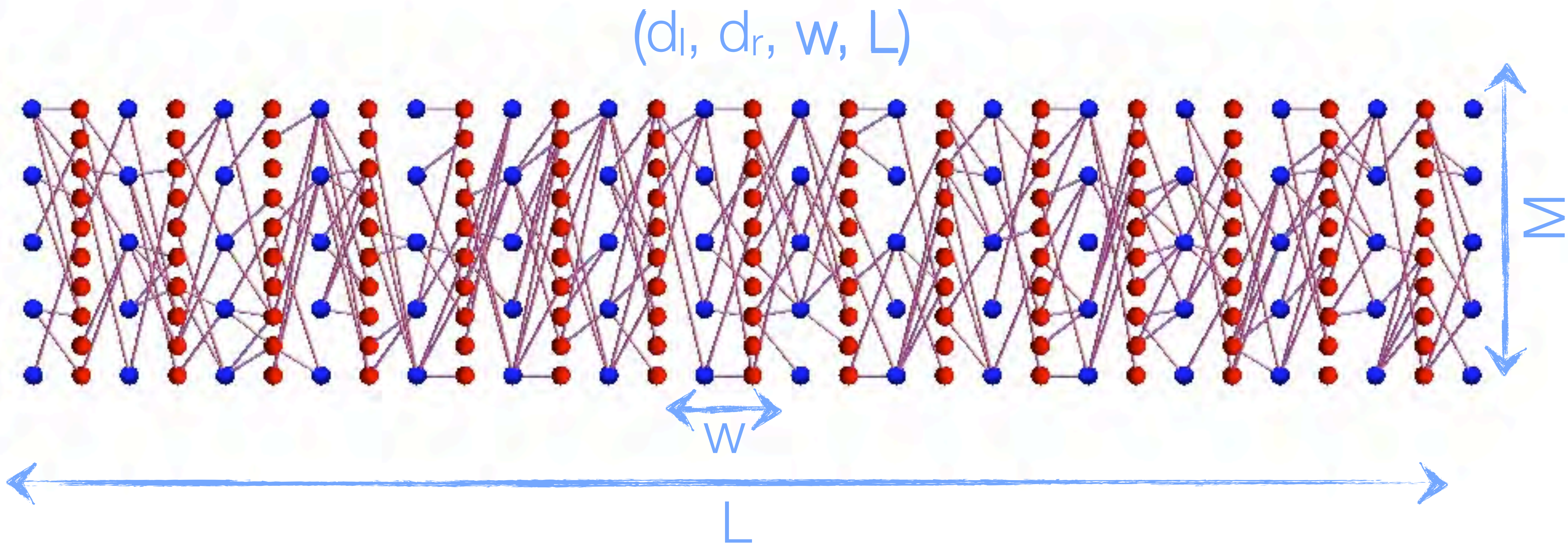
Scaling Exponent for Spatially Coupled Codes



N=L M

$$P_N \sim L\sqrt{M}e^{-\alpha\delta^2 M} \qquad \text{loss due to scaling}$$

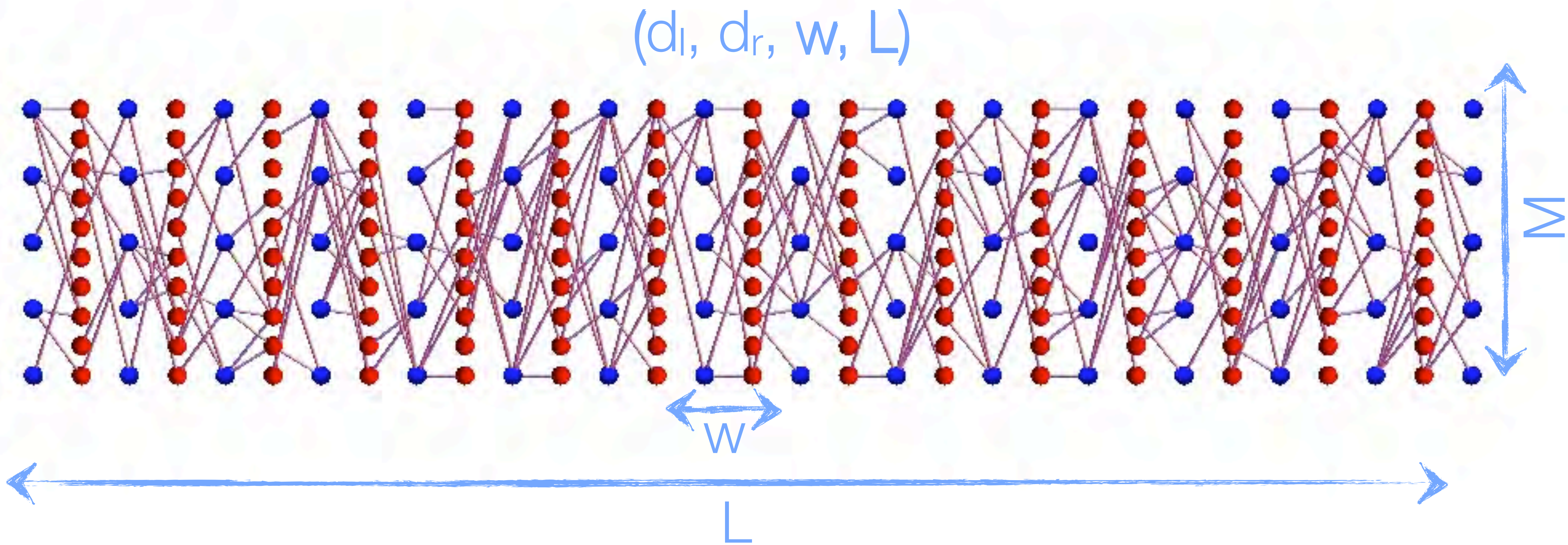
Scaling Exponent for Spatially Coupled Codes



N=L M

$P_N \sim L\sqrt{M}e^{-\alpha\delta^2 M}$ loss due to scaling $\delta = \dfrac{\beta}{\sqrt{M}}$

Scaling Exponent for Spatially Coupled Codes

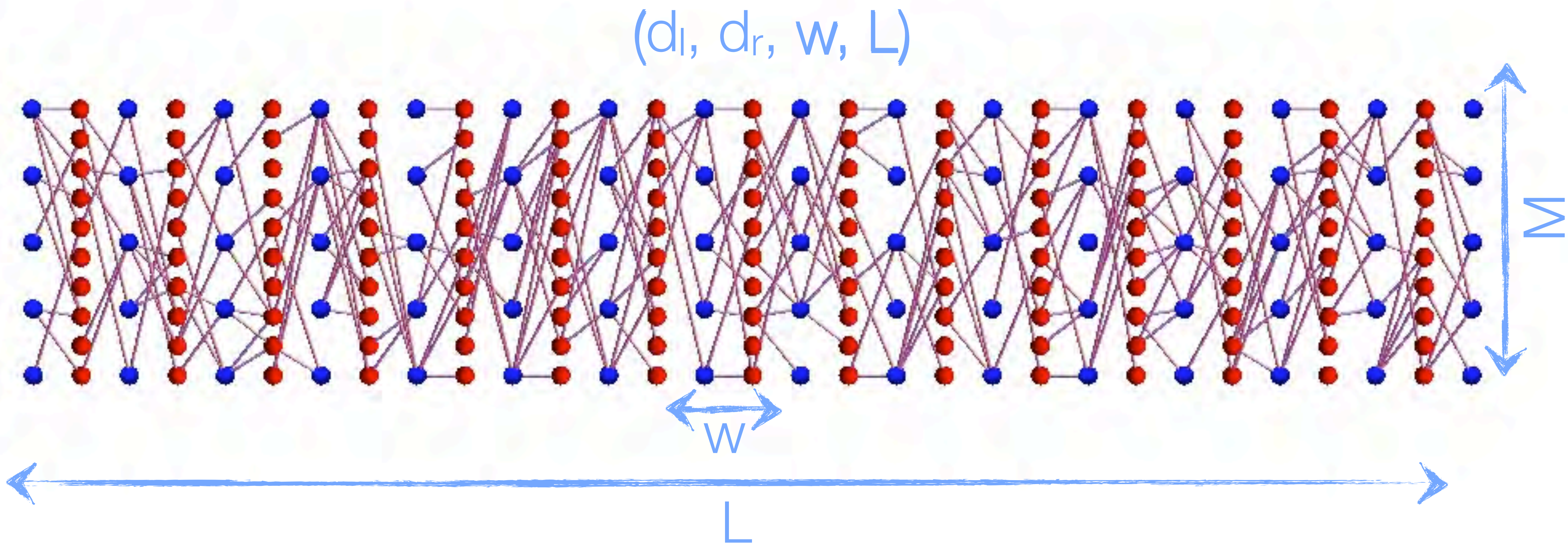


N=L M

$$P_N \sim L\sqrt{M}e^{-\alpha\delta^2 M}$$ loss due to scaling $$\delta = \frac{\beta}{\sqrt{M}}$$

$$R = R_{des} - \frac{1}{L}$$ loss due boundary

Scaling Exponent for Spatially Coupled Codes



N=L M

$P_N \sim L\sqrt{M}e^{-\alpha\delta^2 M}$ loss due to scaling $\delta = \frac{\beta}{\sqrt{M}}$

$R = R_{des} - \frac{1}{L}$ loss due boundary

gap to capacity $= \delta + \frac{1}{L} = \frac{1}{\sqrt{M}} + \frac{1}{L} = N^{-\frac{1}{3}}$

Summary — Scaling Exponent

	optimal	Polar	Spatially Coupled
error exponent	$e^{-NE(R,W)}$	$2^{-\sqrt{N}2^{\frac{\sqrt{\log_2 N}}{2}Q^{-1}(\frac{R}{I(W)})}}$	$e^{-NG(R,W)}$
scaling	$N \text{ is } \Theta(\frac{1}{(I(W)-R)^2})$	$N \text{ is } \Theta(\frac{1}{(I(W)-R)^\mu})$ $3.579 \leq \mu \leq 4.714$	$N \text{ is } \Theta(\frac{1}{(I(W)-R)^3})$
error floor	$P(W) \leq P(W')^{\frac{F(W)}{F(W')}}$	$P(W) \leq P(W')^{\frac{\log Z(W)}{\log Z(W')}}$	$P(W) \leq P(W')^{\frac{G(W)}{G(W')}}$