Quantum Error Correction by Optimal Control

—Concepts, Applications, Perspectives—

Thomas Schulte-Herbrüggen
TU-Munich

includes joint work with

Ville Bergholm, Corey O'Meara, Gunter Dirr,
Philipp Neumann, Florian Dolde, Fedor Jelezko, Jörg Wrachtrup





Scope

Goal (Dynamic Optimal Control Task)

Subject to obeying its eqn. of motion, steer a dynamic system to maximal figure of merit by admissible controls!

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- is *universal*: state-transfer and gate synthesis in closed or open (bilinear) systems.
- is *flexible*: combines all state-of-the-art modules.

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Quantum Systems and Control Theory provides:

- *optimal controls* for implementing error correcting gates experimentally with HIFI
- *symmetry principles* for dissip. state/code engineering (centraliser $\Leftrightarrow$ stabiliser algebra)
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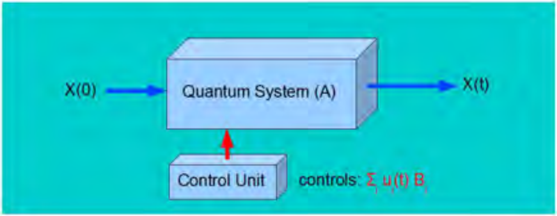
Systems Theory






- Many quantum control systems have common form

$$\dot{X}(t) = -\Big(A + \sum_j u_j(t) B_j\Big) X(t)$$

$X(t)$: 'state', A: drift, B_j: control Hamiltonians, u_j: control amplitudes

Setting and Task	'State' $X(t)$	Drift A	Controls B_j
closed systems:			
pure-state transfer	$X(t) = \lvert\psi(t)\rangle$	iH_0	iH_j
gate synthesis (fixed global phase)	$X(t) = U(t)$	iH_0	iH_j
state transfer	$X(t) = \rho(t)$	$i\widehat{H}_0$	$i\widehat{H}_j$
gate synthesis (free global phase)	$X(t) = \widehat{U}(t)$	$i\widehat{H}_0$	$i\widehat{H}_j$
open systems:			
state transfer	$X(t) = \rho(t)$	$i\widehat{H}_0 + \Gamma$	$i\widehat{H}_j$
quantum-map synthesis	$X(t) = F(t)$	$i\widehat{H}_0 + \Gamma$	$i\widehat{H}_j$

$\widehat{H}$ is Hamiltonian commutator superoperator (generating $\widehat{U}$) in Liouville space.

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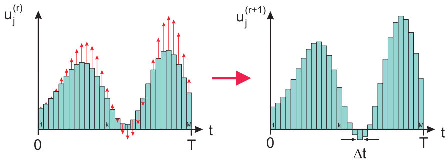
Basic Systems Theory
DYNAMO Platform
Algorithmic Concept
Applications I: Error Correction
Applications II: Fixed-Point Engineering
Applications III: Noise Switching
Conclusions

concurrent (GRAPE) JMR **172** (2005), 296 and PRA **72** (2005), 042331



DYNAMO Platform for Gradient-Based Algorithms

comprises ALL approaches — PRA **84** 022305 (2011)

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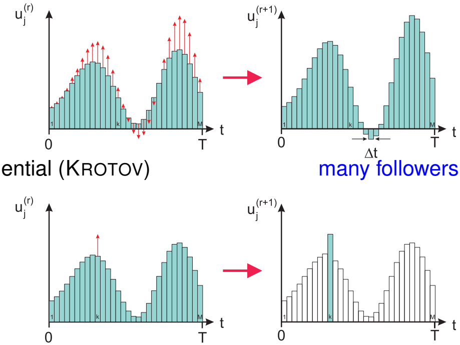
sequential (KROTOV) — many followers of Tannor & Rice (1985)



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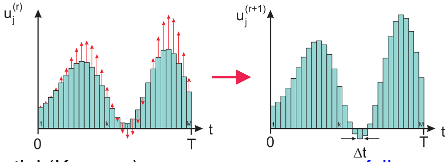

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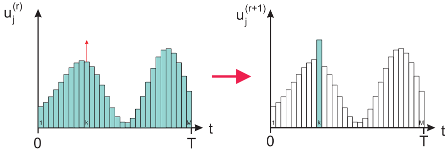
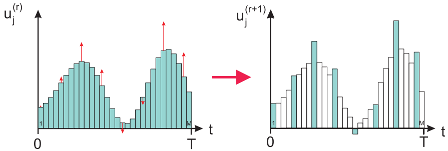
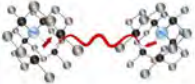


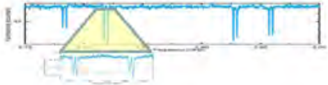
new hybrids PRA **84** (2011), 022305



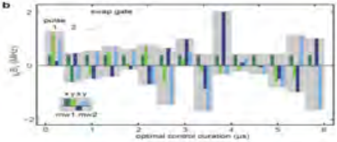
2 coupled NV centres:



- **aim:** high-fideltity production of entangled states
- **challenges:**
  crowded spectrum
  quantization axes not aligned

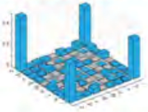


- **optimal-control solution:**



- **high fidelity** entanglement:

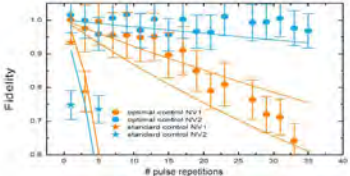
electron spins



f = 82.4 %

nuclear spins

f = 81.9 %



Optimal Control of NV Centres

Error-Correction

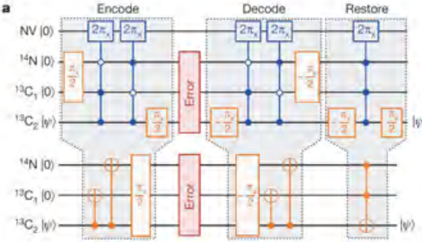
Nature **506**, 204 (2014)

NV centre:

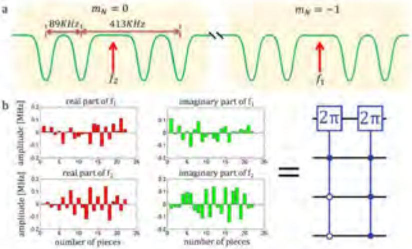
- single-shot read-out

- **hifi** optimal control solution

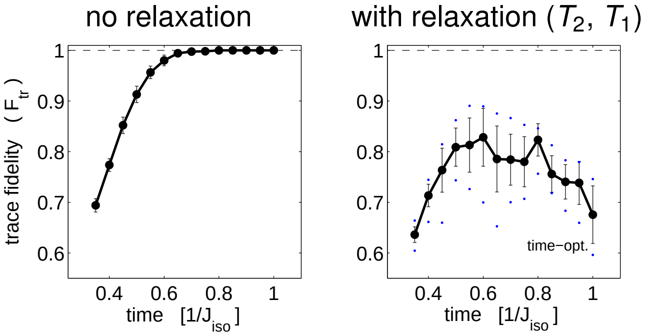
f = 99 %



$f(^{14}N) = 95.8$ %
$f(C_1) = 96.9$ %
$f(C_2) = 99.6$ %



Typical: system drives outside protected subspace



- mean of 15 time-optimised pulse sequences
- dissipation affects sequences differently

J. Phys. B **44** (2011) 154013

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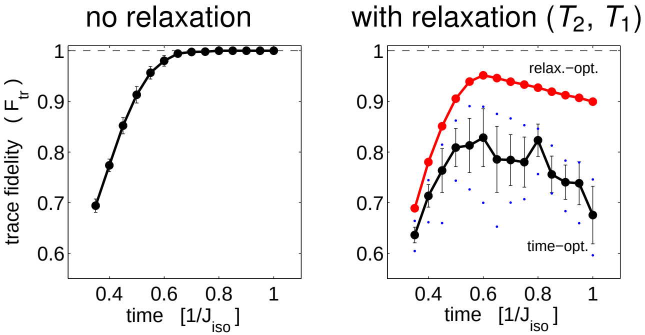
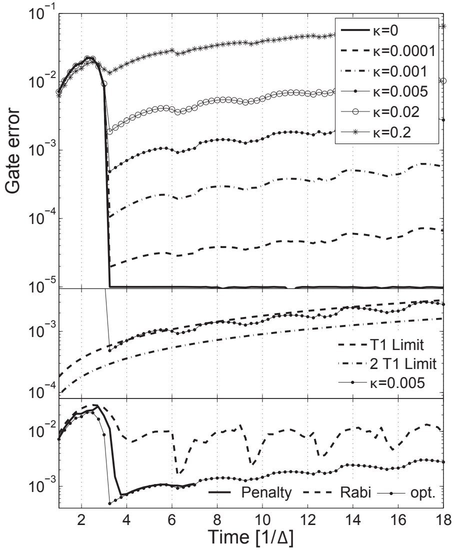


- mean of 15 time-optimised pulse sequences
- dissipation affects sequences differently
- relaxation-optimised: systematic substantial gain

Control of Non-Markovian Open Systems

Qubit Coupled via Two-Level Fluctuator to Spin Bath

with P. Rebentrost and F. Wilhelm



←RABI pulse

←cut error by factor ≤ 10 with optimal control

PRL **102** 090401 (2009)

- Principle: embed to Markovian and project

$$\begin{array}{ccc}
\rho_0 = \rho_{SE}(0) \otimes \rho_B(0) & \xrightarrow{\mathrm{Ad}_{W}(t)} & \rho(t) = W(t)\rho_0 W^\dagger(t) \\
\Pi_{SE} \Big\downarrow \mathrm{tr}_B & & \Pi_{SE} \Big\downarrow \mathrm{tr}_B \\
\rho_{SE}(0) & \xrightarrow[\textit{Markovian}]{F_{SE}(t)} & \rho_{SE}(t) \\
\Pi_{S} \Big\downarrow \mathrm{tr}_E & & \Pi_{S} \Big\downarrow \mathrm{tr}_E \\
\rho_S(0) & \xrightarrow[\textit{non-Markovian}]{F_S(t)} & \rho_S(t)
\end{array}$$

- Gain: relax.-optimised control vs. time-opt. control

category	Markovian	non-Markovian
no encoding: full Liouville space	small–medium	medium–big
encoding: protected subspace	big	difficult[1]

[1] problem roots in finding a viable protected subspace

Devise $\{V_k\}$ such that ρ_∞ is unique global fixed point of

$$\dot\rho = -\Gamma\rho = \sum_k V_k \rho V_k^\dagger - \tfrac{1}{2}\{V_k^\dagger V_k, \rho\}_+$$

1. characterize target fixed-point ρ_∞ by its symmetries: centraliser $\mathfrak{cent}(\rho_\infty) := \{s \,|\, [s, \rho_\infty] = 0\}$
2. determine *max. abelian subalgebra* $\mathfrak{a}$ of $\mathfrak{cent}(\rho_\infty)$
3. pick translations τ according to $\mathfrak{a}$
4. translate into Lindblad terms $\{V_k := \sigma_{\mathbf{p}}^{(k)} + i \cdot \sigma_{\mathbf{q}}^{(k)}\}$ with $\tau_{\mathbf{m}} \mapsto \sigma_{\mathbf{m}} = i\sigma_{\mathbf{p}} \circ \sigma_{\mathbf{q}}$ or $\mathbf{m} = \mathbf{p} \star \mathbf{q}$
5. ensure uniqueness of ρ_∞

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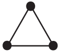
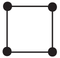
Fixed-Points I

Graph States, Topol. States

Graph	abelian subalgebra $\mathfrak{a}$	$\{\tau_{\mathbf{m}}\}$	$\{V_k\}$
	$\langle xz, zx \rangle$	τ_{xz} τ_{zx}	$V_1 = y1 + \mathrm{i} \cdot zz$ $V_2 = 1y + \mathrm{i} \cdot zz$
	$\langle xz1, zxz, 1zx \rangle$	τ_{xz1} τ_{zxz} τ_{1zx}	$V_1 = y11 + \mathrm{i} \cdot zz1$ $V_2 = 1y1 + \mathrm{i} \cdot zzz$ $V_3 = 11y + \mathrm{i} \cdot 1zz$
	$\langle xzz, zxz, zzx \rangle$	τ_{xzz} τ_{zxz} τ_{zzx}	$V_1 = y11 + \mathrm{i} \cdot zzz$ $V_2 = 1y1 + \mathrm{i} \cdot zzz$ $V_3 = 11y + \mathrm{i} \cdot zzz$
	$\langle xz1z, zxz1, 1zxz, z1zx \rangle$	τ_{xz1z} τ_{zxz1} τ_{1zxz} τ_{z1zx}	$V_1 = y111 + \mathrm{i} \cdot zz1z$ $V_2 = 1y11 + \mathrm{i} \cdot zzz1$ $V_3 = 11y1 + \mathrm{i} \cdot 1zzz$ $V_4 = 111y + \mathrm{i} \cdot z1zz$

Fixed-Points II
More States

Target FP	$\{\tau_{\mathbf{m}}\}$	$\{V_k\}$
ground state	$\tau_{z11..1}$ $\tau_{1z1..1}$ $\dots$	$V_1 = \sigma^+11..1$ $V_2 = 1\sigma^+1..1$ & perms. $\dots$
GHZ state	$\tau_{xx..x}$ $\tau_{zz1..1}$ $\tau_{1zz1..1}$ $\dots$	$V_1 = y1..1 + \mathrm{i} \cdot zx..x$ $V_2 = x11..1 + \mathrm{i} \cdot yz1..1$ $V_3 = 1x1..1 + \mathrm{i} \cdot 1yz..1$ $\dots$
W state	$-\tau_{zz..z}$ $\tau_{z1..1} - \tau_{1z..1}$ $\tau_{1z1..1} - \tau_{11z..1}$ $\dots$	$V_1 = y11 + \mathrm{i} \cdot zzz$ $V_2 = \sigma^+11..1 - 1\sigma^+1..1$ $V_3 = 1\sigma^+11..1 - 11\sigma^+1..1$ $\dots$
Dicke state	$-\tau_{zz..z}$ $\tau_{zz11..1} - \tau_{1z1z..1}$ $\tau_{11zz1..1} - \tau_{11z1z..1}$ $\dots$	$V_1 = y11..1 + \mathrm{i} \cdot zzz..z$ $V_2 = \sigma^+\sigma^+11..1 - \sigma^+1\sigma^+1..1$ $V_3 = 1\sigma^+\sigma^+11..1 - 1\sigma^+1\sigma^+1..1$ $\dots$

Consider the Lindblad control system Σ

$$\dot{\rho} = -\big((i\widehat{H}_0 + \hat{\Gamma}_0) + i\widehat{H}_u\big)\rho \qquad \rho(0) := \rho_0$$

with $\widehat{H}_u := \sum_j u_j(t)\widehat{H}_j$ and $\widehat{\Gamma}_0(\rho) := \sum_k V_k \rho V_k^\dagger - \frac{1}{2}\{V_k^\dagger V_k, \rho\}_+$.

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Embedding I

The system Lie algebra $\mathfrak{g}_\Sigma \subseteq \mathfrak{g}_{LK}$ given as Lie closure

$$\mathfrak{g}_\Sigma := \langle (iH_0 + \Gamma_0), iH_j \,|\, j = 1, \ldots, m \rangle_{\mathrm{Lie}}$$

comprises the Lie wedge $\mathfrak{w}_\Sigma \subseteq \mathfrak{g}_\Sigma$.

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Embedding II

The Lindblad-Kossakowski Lie algebra $\mathfrak{g}_{LK}$ reads

$$\mathfrak{g}_{LK} := \mathfrak{gl}(\mathfrak{her}_{N^2}) \oplus_s \mathfrak{i}_0$$

with $\mathfrak{i}_0 \simeq \mathbb{R}^{N^2}$. It generates a group of affine maps

$$\mathbf{G} := \mathrm{GL}(\mathfrak{her}_{N^2}) \otimes_s I_0 \supseteq \mathbf{T}$$

embracing the Lie-semigroup of LK-quantum maps $\mathbf{T}$.

Algebraic Structure: 2-Qubit Examples I

Lie Wedges and Embedding in System Algebras

Noise	Lindblad-V	Control-H	Drift-H	$\dim(\mathfrak{g}_\Sigma)$	$\dim(\mathfrak{w}_\Sigma - \mathfrak{w}_\Sigma)$
unital	$(y,z)1$	x1,1x	$z1{+}1z{+}zz$	225	11
deph.	$z1$	$x1$	–”–	22	6
–”–	–”–	$1x$	–”–	5	4
bit-flip	$x1$	$x1$	–”–	16	4
–”–	–”–	$1x$	–”–	52	4
unital	$(y,z)1$	x1,1x	$z1{+}1z{+}H_{xxx}$	225	12
deph.	$z1$	$x1$	–”–	225	6
–”–	–”–	$1x$	–”–	225	4
bit-flip	$x1$	$x1$	–”–	124	4
–”–	–”–	$1x$	–”–	225	4

Noise	Lindblad-V	Control-H	Drift-H	$\mathfrak{g}_\Sigma$	$\dim(\mathfrak{w}_\Sigma - \mathfrak{w}_\Sigma)$
deph.	$z1, 1z$	$\mathfrak{su}(4)$	$z1{+}1z{+}zz$	$\mathfrak{g}_0^{LK}$	135
–"–	$z1, 1z$	$\mathfrak{su}(2) \oplus \mathfrak{su}(2)$	–"–	$\mathfrak{g}_0^{LK}$	21
–"–	$z1, 1z, zz$	$\mathfrak{su}(2) \oplus \mathfrak{su}(2)$	–"–	$\mathfrak{g}_0^{LK}$	27
deph.	$z1, 1z$	$x1, 1x$	–"–	$\mathfrak{g}_0^{LK}$	14
depol.	iso_2	$\mathfrak{su}(4)$	–"–	$\widehat{\mathfrak{su}}(4){+}\mathbb{R}\Gamma$	16
–"–	$\text{iso}_{1:1}$	$\mathfrak{su}(2) \oplus \mathfrak{su}(2)$	–"–	$\widehat{\mathfrak{su}}(2) \oplus \widehat{\mathfrak{su}}(2){+}\mathbb{R}\Gamma$	7
amp.	$+1, 1+$	$\mathfrak{su}(4)$	–"–	$\mathfrak{g}^{LK}$	...
damp.	$+1, 1+$	$\mathfrak{su}(2) \oplus \mathfrak{su}(2)$	–"–	$\mathfrak{g}^{LK}$	...

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pure-state transfer	$X(t) = \vert\psi(t)\rangle$	iH_0	iH_j
gate synthesis (fixed global phase)	$X(t) = U(t)$	iH_0	iH_j
state transfer	$X(t) = \rho(t)$	$i\widehat{H}_0$	$i\widehat{H}_j$
gate synthesis (free global phase)	$X(t) = \widehat{U}(t)$	$i\widehat{H}_0$	$i\widehat{H}_j$
open systems:			
state transfer I	$X(t) = \rho(t)$	$i\widehat{H}_0 + \Gamma$	$i\widehat{H}_j$
quantum-map synthesis	$X(t) = F(t)$	$i\widehat{H}_0 + \Gamma$	$i\widehat{H}_j$

$\widehat{H}$ is Hamiltonian commutator superoperator (generating $\widehat{U} := U(\cdot)U^\dagger$) in Liouville space.

$$\dot{X}(t) = -\Big(A + \sum_j u_j(t) B_j\Big) X(t)$$

$X(t)$: 'state'; A: drift; B_j: control Hamiltonians; u_j: control amplitudes

Setting and Task	'State' $X(t)$	Drift A	Controls B_j
closed systems:			
pure-state transfer	$X(t) = \lvert\psi(t)\rangle$	iH_0	iH_j
gate synthesis (fixed global phase)	$X(t) = U(t)$	iH_0	iH_j
state transfer	$X(t) = \rho(t)$	$i\widehat{H}_0$	$i\widehat{H}_j$
gate synthesis (free global phase)	$X(t) = \widehat{U}(t)$	$i\widehat{H}_0$	$i\widehat{H}_j$
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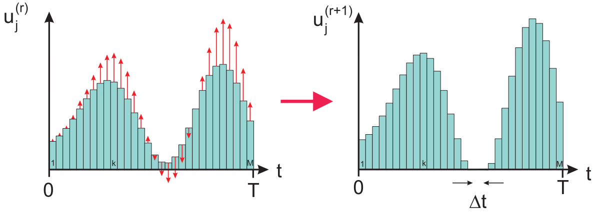
quantum-map synthesis	$X(t) = F(t)$	$i\widehat{H}_0 + \Gamma$	$i\widehat{H}_j$
state transfer II	$X(t) = \rho(t)$	$i\widehat{H}_0$	$i\widehat{H}_j, \Gamma_j$

$\widehat{H}$ is Hamiltonian commutator superoperator (generating $\widehat{U} := U(\cdot)U^\dagger$) in Liouville space.

- add switchable noise amplitudes as further controls



switchable amp-damp noise: $\gamma(t) \cdot \Gamma_L$ with $V_a := \mathbb{1} \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ in

$$\Gamma_L(\rho) = \tfrac{1}{2}\{V_a^\dagger V_a, \rho\}_+ - V_a \rho V_a^\dagger$$

Theorem ('woodcut' version)

Let Σ_a be an n-spin-$\frac{1}{2}$ ZZ-coupled unitarily controllable system.
Adding bang-bang switchable ($\gamma(t) \in [0, 1]$) amp-damp noise on 1 spin allows that any target state can be reached from any initial state

$$\overline{\text{Reach}_{\Sigma_a}(\rho_0)} = \{\textit{all density ops.}\} \quad \textit{for all } \rho_0 \ .$$

switchable bit-flip noise: $\gamma(t) \cdot \Gamma_L$ with $V_b := \mathbb{1} \otimes \sigma_x/2$ in

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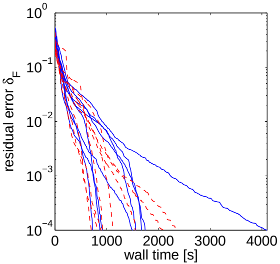
Theorem ('woodcut')

Let Σ_a be an n-spin-$\frac{1}{2}$ ZZ-coupled unitarily controllable system.
Adding bang-bang switchable bit-flip noise on 1 spin allows that any target state majorised by the initial state can be reached

$$\overline{\text{Reach}_{\Sigma_b}(\rho_0)} = \{\rho \,|\, \rho \prec \rho_0\} \quad \textit{for all } \rho_0 .$$

Example

- system: 3-qubit Ising-*ZZ* chain, x, y-controls, controllable noise on terminal qubit
- task I: rand $\rho_0 \rightarrow \rho_{\rm tar}$ by amp-damp
- task II: rand $\rho_0 \rightarrow \rho_{\rm tar} \prec \rho_0$ by bit flip

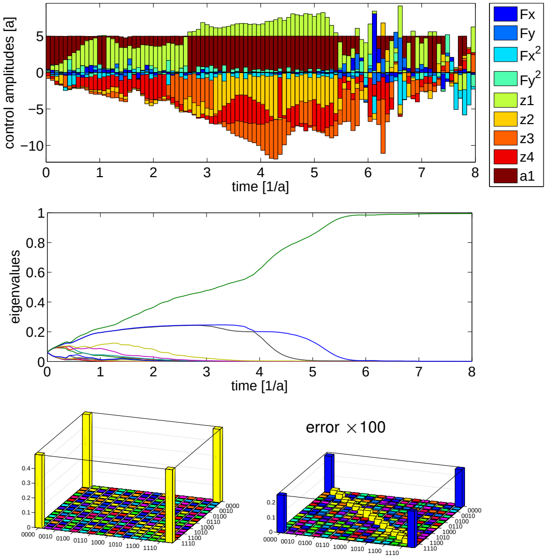


Example

- system: 4-ion system, individual z-controls, joint F_x, F_y-controls, joint $(F_x)^2, (F_y)^2$-controls, and controllable amp-damp noise on terminal qubit
- task III: $\rho_0 \simeq \mathbb{1} \;\rightarrow\; \rho_{|GHZ_4\rangle}$ by amp-damp

Noise-Driven State Transfer III: Ion Traps

arXiv:1206.4945

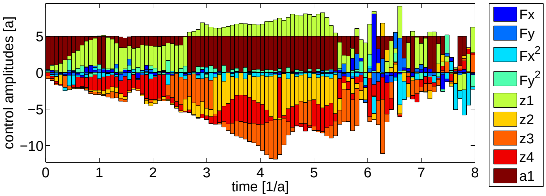


Noise-Driven State Transfer III: Ion Traps

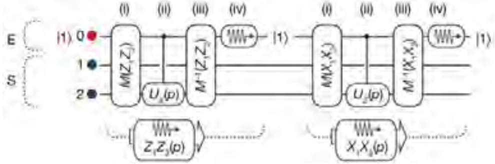
Transfer to GHZ State

arXiv:1206.4945

- open-loop noise control



- may replace measurement-based closed loop feedback



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Markovian vs. non-Markovian State Transfer

For state transfer, Markovian quantum maps are as powerful as non-Markovian maps, i.e. closed-loop control can be replaced by open-loop control.

Conclusions

HIFI quantum engineering for closed and open systems.

- DYNAMO platform
  - optimised gates: enabling HIFI error correction
- symmetry principles of fixed-point engineering
  - centraliser (stabiliser)
- Is open-loop coherent control + *switchable* Markov noise as strong as closed-loop control ?
  - yes for state transfer
  - no for gate/map synthesis

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F. Dolde, P. Neumann, F. Jelezko, J. Wrachtrup

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IEEE TAC **57**, 2050 (2012); arXiv:1206.4945;
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$$\Gamma_L(\rho) = \tfrac{1}{2}\{V_a^\dagger V_a, \rho\}_+ - V_a \rho V_a^\dagger$$

Theorem

Let Σ_a be an n-qubit bilinear control system satisfying (WH) for $\gamma = 0$. Suppose the amp-damp noise amplitude can be switched $\gamma(t) \in \{0, \gamma_*\}$ with $\gamma_* > 0$. If H_d is diagonal (Ising-ZZ type) and the only drift term, then Σ_a acts transitively on the set of all density operators $\mathfrak{pos}_1$

$$\overline{\text{Reach}_{\Sigma_a}(\rho_0)} = \mathfrak{pos}_1 \quad \textit{for all } \rho_0 \in \mathfrak{pos}_1$$

where the closure is understood as the limit $T\gamma_* \to \infty$.

Proof.

- choose diagonal $\rho_0 =: \text{diag}\,(r_0)$
- with H_d diagonal (Ising-ZZ), evolution remains diagonal

$$r(t) = \left[\mathbb{1}_2^{\otimes(n-1)} \otimes \begin{pmatrix} 1 & 1-\epsilon \\ 0 & \epsilon \end{pmatrix}\right] r_0 \text{ with } \epsilon := e^{-t\gamma_*}$$

- can obtain any state

$$\rho(t) = \text{diag}\,(\ldots, [\rho_{ii} + \rho_{jj} \cdot (1-\epsilon)]_{ii}, \ldots, [\rho_{jj} \cdot \epsilon]_{jj}, \ldots);$$

- in limit $T\gamma_* \to \infty$ obtain set of all diagonal density operators $\Delta \subset \mathfrak{pos}_1$;
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Reachable Sets I: Non-Unital

Controlled Amplitude Damping Noise

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- with $2^{n-1} - 1$ switches all but one desired transfer remain;
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Proof.

- again choose diagonal $\rho_0 \;=:\; \mathrm{diag}\,(r_0)$
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$$r(t) = \left[\mathbb{1}_2^{\otimes(n-1)} \otimes \tfrac{1}{2}\begin{pmatrix}(1+\epsilon) & (1-\epsilon)\\ (1-\epsilon) & (1+\epsilon)\end{pmatrix}\right] r_0 \text{ with } \epsilon := e^{-\frac{t}{2}\gamma_*}$$

- NB: $\rho_{\text{tar}} \prec \rho_0$ iff $\rho_{\text{tar}} = D\rho_0$ with doubly stochastic D product of at most $N-1$ such T-transforms
  (e.g., Thm. B.6 in MARSHALL-OLKIN or Thm. II.1.10 in BHATIA)
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Reachable Sets II: Unital

Controlled Bit-Flip Noise

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- to limit relaxative averaging to first two eigenvalues, conjugate ρ_0 with $U_{12} := \mathbb{1}_2 \oplus \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}^{\oplus 2^{n-1}-1}$
- gives protected state $\rho_0' := U_{12}\rho_0 U_{12}^\dagger$

$$\rho_0' = \begin{pmatrix} \rho_{11} & 0 \\ o & \rho_{22} \end{pmatrix} \oplus \tfrac{1}{2} \begin{pmatrix} \rho_{33}+\rho_{44} & \rho_{33}-\rho_{44} \\ \rho_{33}-\rho_{44} & \rho_{33}+\rho_{44} \end{pmatrix} \oplus \cdots$$

- now relaxation acts as T-transform on ρ_0'
- NB: … iff … with doubly stochastic D

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$$\rho(t) = \text{diag}\,\Big(\ldots, \tfrac{1}{2}[\rho_{ii} + \rho_{jj} + (\rho_{ii} - \rho_{jj}) \cdot e^{-\frac{t}{2}\gamma_*}]_{ii}, \ldots \\ \tfrac{1}{2}[\rho_{ii} + \rho_{jj} + (\rho_{jj} - \rho_{ii}) \cdot e^{-\frac{t}{2}\gamma_*}]_{jj}, \ldots \Big)$$

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- in limit $T\gamma_* \to \infty$ obtain set of all diagonal density

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$$r(t) = \left[\mathbb{1}_2^{\otimes(n-1)} \otimes \tfrac{1}{2}\begin{pmatrix}(1+\epsilon) & (1-\epsilon)\\(1-\epsilon) & (1+\epsilon)\end{pmatrix}\right] r_0 \text{ with } \epsilon := e^{-\frac{t}{2}\gamma_*}$$

- NB: $\rho_{\text{tar}} \prec \rho_0$ iff $\rho_{\text{tar}} = D\rho_0$ with doubly stochastic D product of at most $N-1$ such T-transforms

  (e.g., Thm. B.6 in MARSHALL-OLKIN or Thm. II.1.10 in BHATIA)

- in limit $T\gamma_* \to \infty$ obtain set of all diagonal density operators $\mathrm{diag}\,(r) \prec \mathrm{diag}\,(r_0)$
- by unitary controllability get all density operators $\rho \prec \rho_0$.

□

Proof: further details.

- decouple protected states ρ_0' from Hamiltonian H_0
- to this end, observe

$$e^{i\pi H_{1x}} e^{-t(\Gamma+iH_{zz})} e^{-i\pi H_{1x}} = e^{-t(\Gamma-iH_{zz})}$$

- so decoupling obtained in Trotter limit

$$\lim_{k\to\infty} \left(e^{-\frac{t}{2k}(\Gamma+iH_{zz})} e^{-\frac{t}{2k}(\Gamma-iH_{zz})}\right)^k = e^{-t\Gamma} \, .$$

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- *T*-transformation is convex combination $\lambda \mathbb{1} + (1-\lambda) Q$ with pair transposition Q and $\lambda \in [0,1]$
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Proof: further details.

one cannot go beyond states majorised by ρ_0:

- bit-flip superoperator: doubly-stochastic

$$e^{-t\Gamma_b} = \mathbb{1}_4^{\otimes(n-1)} \otimes \frac{1}{2}\begin{pmatrix} (1+\epsilon) & 0 & 0 & (1-\epsilon) \\ 0 & (1+\epsilon) & (1-\epsilon) & 0 \\ 0 & (1-\epsilon) & (1+\epsilon) & 0 \\ (1-\epsilon) & 0 & 0 & (1+\epsilon) \end{pmatrix}$$

- bit-flip plus unitary control: cpt unital map hence also generalised doubly-stochastic linear map Φ in sense of ANDO, *Lin. Alg. Appl.* **118** (1989) p 235 Thm. 7.1 saying that for any hermitian A: $\Phi(A) \prec A$.

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new control term: $\gamma(t) \cdot \Gamma_L$ with $V_\theta := \begin{pmatrix} 0 & (1-\theta) \\ \theta & 0 \end{pmatrix}, \theta \in [0,1]$ in

$$\Gamma_L(\rho) = \tfrac{1}{2}\{V_\theta^\dagger V_\theta, \rho\}_+ - V_\theta \rho V_\theta^\dagger$$

- fixed point (single qubit)
  $\rho_\infty(\theta) = \frac{1}{\bar\theta^2 + \theta^2} \begin{pmatrix} \bar\theta^2 & 0 \\ 0 & \theta^2 \end{pmatrix}$ with $\bar\theta := 1 - \theta$
- compare with canonical density operator at temperature β
  $\rho_\beta := \frac{1}{2\cosh(\beta/2)} \begin{pmatrix} e^{\beta/2} & 0 \\ 0 & e^{-\beta/2} \end{pmatrix}$
- so θ relates to inverse temperature $\beta(\theta) := \frac{1}{k_B T_\theta}$ by
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Theorem

Let Σ_θ be an n-qubit bilinear control system satisfying (WH) for $\gamma = 0$. Suppose the V_θ noise amplitude can be switched $\gamma(t) \in \{0, \gamma_*\}$. If all drift components of H_d are diagonal (Ising-ZZ), then Σ_θ gives for the thermal state $\rho_0 = \frac{1}{2^n}\mathbb{1}$

$$\overline{\text{Reach}}_{\Sigma_\theta}(\tfrac{1}{2^n}\mathbb{1}) \supseteq \{\rho \,|\, \rho \prec \rho_\delta\}$$

where ρ_δ is the purest state obtainable by partner-pairing algorithmic cooling with bias $\delta := \frac{\bar{\theta}^2 - \theta^2}{\bar{\theta}^2 + \theta^2}$ (again closure by $T\gamma_* \to \infty$).

Theorem (Pontryagin)

Consider a system governed by $\dot{X}(t) = F(X, u, t)$. For $u_*(t)$ to be an optimal control steering $X(0)$ into $X(T)$ so that $J[X(t)] = \int_0^T L(t)dt$ assumes its critical points over (almost) all times, it suffices there is

- *an adjoint system $\lambda(t)$ satisfying $\dot{\lambda} = -\frac{\partial h}{\partial X}$ by virtue of*
- *a scalar Hamiltonian function (so $\dot{X}(t) \equiv F(X, u, t) = \frac{\partial h}{\partial \lambda^\dagger}$), $h(P, X, u, t) := L(X, u, t) + \langle \lambda(t) | F(X, u, t) \rangle$ where*
    - *h attains its critical points for optimal controls $u_*(t)$, i.e., $\frac{\partial h}{\partial u_*(t)} = 0$ at almost all $0 \leq t \leq T$;*
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Pontryagin's Maximum Principle

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Maximum Principle, ctd

Proof.

- FRÉCHET derivatives provide $\frac{\partial L}{\partial X} \in \text{Mat}_n(\mathbb{C})$ and $\frac{\partial L}{\partial u} \in \text{Mat}_{n,1}(\mathbb{C})$.
- Thus for $J[X(t)] = \int_0^T dt L(X, u, t)$ calculate first variation in X and u as

$$\delta J \stackrel{1^\circ}{=} J(X + \delta X, u + \delta u, t) - J(X, u, t)$$
$$= \int_0^T dt \{ \langle \frac{\partial L}{\partial X} | \delta X \rangle + \langle \frac{\partial L}{\partial u} | \delta u \rangle \} + L(t) \delta t \Big|_0^T \quad .$$

NB: δX depends on variation of control δu via $\dot{X} = F(X, u, t)$.

- Incorporate dependence of δX on δu as in eqn. of motion by *operator-valued* LAGRANGE *multiplier* $\lambda(t)$ associated with zero-cost term

$$J_\lambda := \int_0^T dt \langle \lambda(t) | F(X, u, t) - \dot{X} \rangle = 0 \quad .$$

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Maximum Principle, ctd

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Maximum Principle, ctd

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Basic Systems Theory
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Maximum Principle, ctd

- Introduce scalar-valued Hamiltonian function

$$h(X,\lambda,u,t) := L(X,u,t) + \langle \lambda(t) | F(X,u,t) \rangle = L + \langle \lambda | \dot{X} \rangle$$

to finally arrive at

$$\delta J + \delta J_\lambda = \int_0^T dt \Big\{ \langle \frac{\partial h}{\partial X} + \dot{\lambda} | \delta X \rangle + \langle \frac{\partial h}{\partial u} | \delta u \rangle \Big\} + h(X,\lambda,u,T)\delta t \,.$$

- Therefore optimal controls $u_*(t)$ leading to quality-optimising trajectories $X_*(t)$ and their adjoints $\lambda_*(t)$ result if

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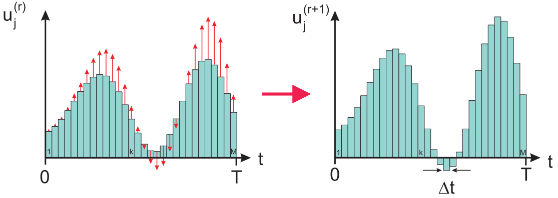
Getting Optimized Quantum Controls

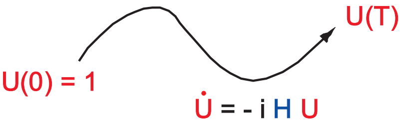
Gradient Flow on Control Amplitudes

- Gradient Assisted Pulse Engineering GRAPE

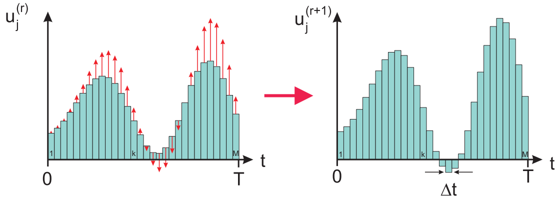


J. Magn. Reson. **172** (2005), 296 and *Phys. Rev. A* **72** (2005), 042331

Generation of Unitary Operators



Gradient Assisted Algorithm GRAPE



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$$h(U) = \mathrm{Re}\,\mathrm{tr}\{\lambda^\dagger(-i(H_d + \textstyle\sum_j u_j H_j))U\}$$

2. with adjoint system satisfying

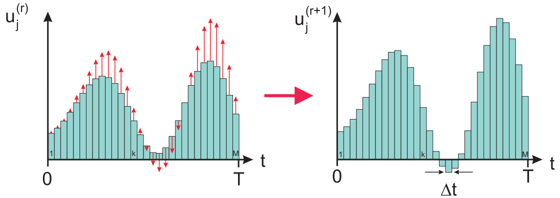
$$\dot\lambda(t) = -i(H_d + \textstyle\sum_j u_j H_j)\lambda(t) \quad .$$

3. Then PONTRYAGIN's maximum principle requires

$$\frac{\partial h}{\partial u_j} = \mathrm{Re}\,\mathrm{tr}\{\lambda^\dagger(-iH_j)U\} \overset{!}{=} 0$$

4. thus allowing for a gradient-flow of quantum controls

Gradient Assisted Algorithm GRAPE



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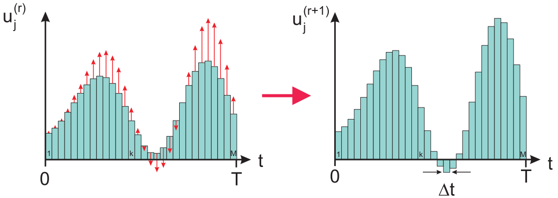

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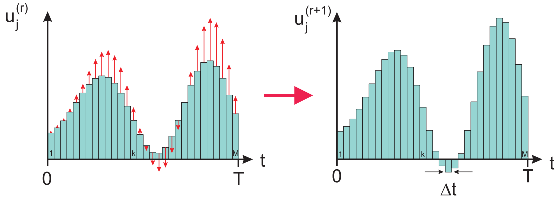

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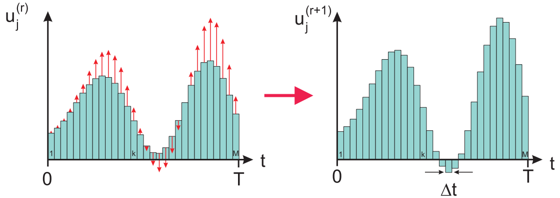

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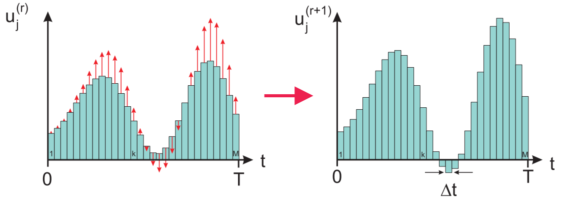
Gradient Assisted Algorithm GRAPE



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- Viewing Markovian Quantum Channels as Lie Semigroups with GKS-Lindblad Generators as Lie Wedge

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Markoviantity, Divisibility I
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GKS-Lindblad Gen.
Divisibility II

Observe: two notions

Definition

- A CP-Map T is *(infinitely) divisible*, if $\forall r \in \mathbb{N}$ there is a S with $T = S^r$.
- A CP-map T is *infinitesimally divisible* if $\forall \epsilon > 0$ there is a sequence $\prod_{j=1}^{r} S_j = T$ with $||S_j - \mathrm{id}|| \leq \epsilon$.

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Notions:
time-(in)dependent CP-map: solution of *time-(in)dependent* master eqn. $\dot{X} = -\mathcal{L} \circ X$.

Theorem (Wolf & Cirac (2008))

- *The set of all time-independent Markovian CP-maps coincides with the set of all (infinitely) divisible CP-maps.*
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Observe: semigroup structure

$$\text{Reach}(\mathbb{1}, t_1) \circ \text{Reach}(\mathbb{1}, t_2) = \text{Reach}(\mathbb{1}, t_1 + t_2)\ \forall t_\nu \geq 0$$

Definition

- A *subsemigroup* $\mathbf{S} \subset \mathbf{G}$ of a Lie group $\mathbf{G}$ with algebra $\mathfrak{g}$ contains $\mathbb{1}$ and follows $\mathbf{S} \circ \mathbf{S} \subseteq \mathbf{S}$. Its largest subgroup is denoted $E(\mathbf{S}) := \mathbf{S} \cap \mathbf{S}^{-1}$.
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Definition (Lie Wedge and Lie Semialgebra)

- A *wedge* $\mathfrak{w}$ is a closed convex cone of a finite-dim. real vector space.
- Its *edge* $E(\mathfrak{w}) := \mathfrak{w} \cap \text{-}\mathfrak{w}$ is the largest subspace in $\mathfrak{w}$.
- It is a *Lie wedge* if it is invariant under conjugation

$$e^{\mathrm{ad}_g}(\mathfrak{w}) \equiv e^{g} \mathfrak{w} e^{-g} = \mathfrak{w}$$

  for all edge elements $g \in E(\mathfrak{w})$.
- A *Lie semialgebra* is a Lie wedge compatible with BCH multiplication $X * Y := X + Y + \frac{1}{2}[X, Y] + \dots$ so that for a BCH neighbourhood B of $0 \in \mathfrak{g}$

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Define as completely positive, trace-preserving invertible linear operators the set $\mathbf{P}^{\mathbf{cp}}$, and let $\mathbf{P}_0^{cp}$ denote the connected component of the unity.

Theorem (Kossakowski, Lindblad)

The Lie wedge to the connected component of the unity of the semigroup of all invertible CPTP maps is given by the set of all linear operators of GKS-Lindblad form:

$$
\begin{aligned}
L(\mathbf{P}_0^{cp}) &= \{-\mathcal{L} \,|\, \mathcal{L} = -(i\,\mathrm{ad}_H + \Gamma_L)\} \quad \textit{with} \\
\Gamma_L(\rho) &= \tfrac{1}{2} \sum_k \{V_k^\dagger V_k, \rho\}_+ - 2 V_k \rho V_k^\dagger
\end{aligned}
$$

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Theorem

The semigroup

$$\mathbf{T} := \overline{\langle \exp\left(\mathrm{L}(\mathbf{P}_0^{\mathrm{cp}})\right)\rangle}_S \subseteq \mathbf{P}_0^{\mathrm{cp}}$$

generated by $\mathrm{L}(\mathbf{P}_0^{\mathrm{cp}})$ *is a Lie subsemigroup with global Lie wedge* $\mathrm{L}(\mathbf{T}) = \mathrm{L}(\mathbf{P}_0^{\mathrm{cp}})$.

Corollary (to Wolf, Cirac (2008))

$\mathbf{P}_0^{\mathrm{cp}}$ *itself is not a Lie subsemigroup, yet it comprises*

(1) *the set of time independent Markovian channels, i.e. the union of all one-parameter Lie semigroups* $\{\exp(-\mathcal{L}t) \mid t \geq 0\}$ *with* $\mathcal{L}$ *in GKS-Lindblad form;*

(2) *the closure of the set of time dependent Markovian channels, i.e. the Lie semigroup* $\mathbf{T}$*;*

(3) *a set of non-Markovian channels whose intersection with* $\mathbf{P}_0^{\mathrm{cp}}$ *has non-empty interior.*

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GKS-Lindblad Operators as Lie Wedge

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Corollary

A quantum channel is time dependent Markovian iff it allows for a representation $T = \prod_{j=1}^{r} S_j$, *where* $S_1 = e^{-\mathcal{L}_1}, S_2 = e^{-\mathcal{L}_2}, \dots, S_r = e^{-\mathcal{L}_r}$ *so that there is a global Lie wedge* $\mathfrak{w}_r$ *generated by* $\mathcal{L}_1, \mathcal{L}_2, \dots, \mathcal{L}_r$.

Corollary

Let $T = \prod_{j=1}^{r} S_j$ be a time dependent Markovian channel with $S_1 = e^{-\mathcal{L}_1}, S_2 = e^{-\mathcal{L}_2}, \dots, S_r = e^{-\mathcal{L}_r}$ and let $\mathfrak{w}_r$ denote the smallest global Lie wedge generated by $\mathcal{L}_1, \mathcal{L}_2, \dots, \mathcal{L}_r$. Then

- *T boils down to a time independent Markovian channel, if it is sufficiently close to the unity and if there is a representation so that the associated Lie wedge $\mathfrak{w}_r$ specialises to a Lie semialgebra.*

Complements recent work: Wolf,Cirac, *Commun. Math. Phys.* (2008) & Wolf,Eisert,Cubitt,Cirac, *PRL* (2008)

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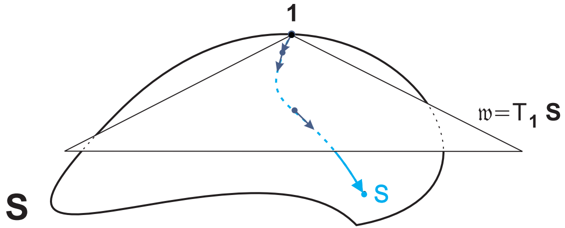
Consider: controlled system with *time dep* Liouvillians $\{\mathcal{L}_u(t)\}$

$$\dot{X} = -\mathcal{L}_u(t)X = -(iH_d + i\sum_j u_j(t)H_j + \Gamma)X$$



Liouvillians $\mathcal{L}_u$ form

- *Lie wedge* $\mathfrak{w}$
- *Lie semialgebra* $\mathfrak{s} \subset \mathfrak{w}$ if $\{\mathcal{L}_u\}$ BCH compatible with $\mathfrak{w}$
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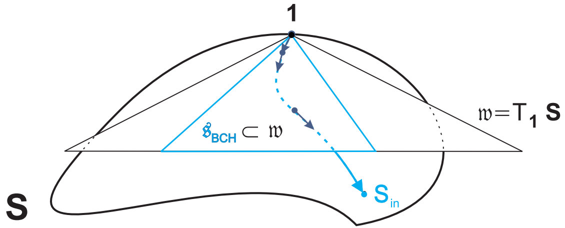

Liouvillians $\mathcal{L}_u$ form $\quad L_j * L_k := L_j + L_k + \frac{1}{2}[L_j, L_k] + \cdots \in \mathfrak{w}$

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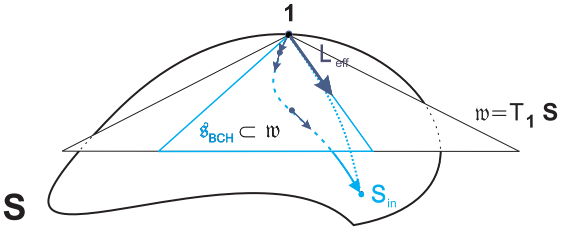

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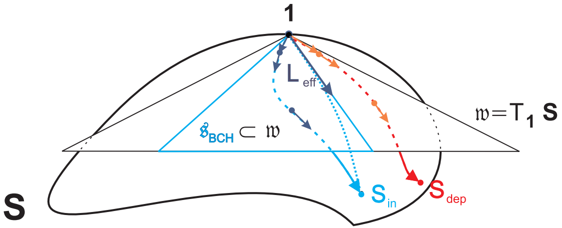

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Rep. Math. Phys. **64** 93 (2009)

Lindbladians Generate (Lie) Semigroup

Consider: controlled system with *time dep* Lindbladians $\{\mathcal{L}_u(t)\}$

$$\dot{X} = -\mathcal{L}_u(t)X = -(iH_d + i\sum_j u_j(t)H_j + \Gamma)X$$

Lindbladians $\{\mathcal{L}_u\}$ form

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Consider bilinear control systems

$\dot{X} = -(A + \sum_j u_j B_j)X$ with $A := i\widehat{H}_d + \Gamma_L$ and $B_j := i\widehat{H}_j$
with system algebras $\mathfrak{g} := \langle A, B_j \,|\, j = 1, 2, \ldots m \rangle_{\text{Lie}}$.

- controllability condition for closed systems:
  $\langle iH_d, iH_j \,|\, j = 1, 2, \ldots m \rangle_{\text{Lie}} = \mathfrak{su}(N)$
- WH-condition for open systems:
  $\langle iH_d, iH_j \,|\, j = 1, 2, \ldots m \rangle_{\text{Lie}} = \mathfrak{su}(N)$
- H-condition for open systms:
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- closed controllable systems:
  Reach $\rho_0 = \mathcal{O}_U(\rho_0) := \{U\rho_0 U^\dagger \mid U \in SU(N)\}$
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IEEE TAC **57**, 2050 (2012)

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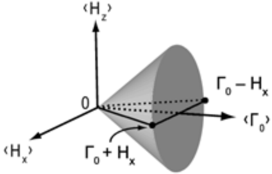
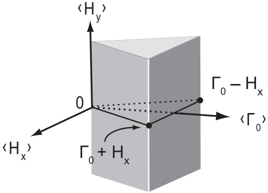
Exploring Reachable Directions

by Lie Wedges to Unital Channels — IEEE TAC **57**, 2050 (2012)

Bilinear control system: $\dot{X} = -(A + \sum_j u_j B_j) X$

- satisfies WH-condition with :
  $A := H_z + \Gamma_0$, $B := uH_y$, and $\Gamma_0 := \operatorname{diag}(1, 0, 1)$
- Lie wedge:
  $$\mathfrak{w}_0 = \langle H_y \rangle \oplus -\mathbb{R}_0^+ \operatorname{conv} \left\{ \begin{bmatrix} \sin(\theta) \\ \cos(\theta) \\ 1 \end{bmatrix} \cdot \begin{bmatrix} H_x \\ H_z \\ \Gamma_0 \end{bmatrix} \;\middle|\; \theta \in \mathbb{R} \right\}$$





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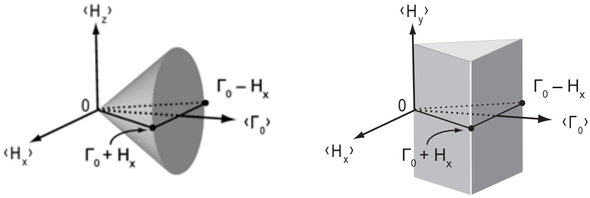

- satisfy WH-condition with :
  $A := H_z + \Gamma_0$, $B := uH_y$, and $\Gamma_0 := \mathrm{diag}\,(1,1,2)$
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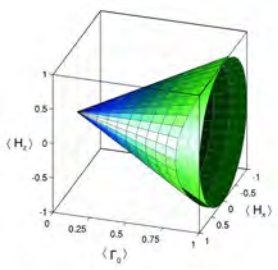

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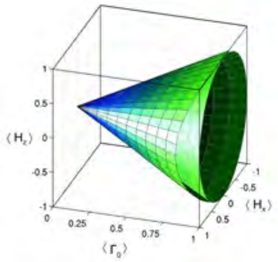

Bilinear control system: $\dot{X} = -(A + \sum_j u_j B_j)X$

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  $A := H'_z + \Gamma'_0$, $B := uH'_y$, and $\Gamma_0 := \mathrm{diag}\,(1, 0, 1)$
- Lie wedge:
  $$\mathfrak{w}_0 = \langle H'_y \rangle \oplus -\mathbb{R}_0^+ \,\mathrm{conv}\left\{ \begin{bmatrix} \sin(\theta) \\ \cos(\theta) \\ 1 \end{bmatrix} \cdot \begin{bmatrix} H'_x \\ H'_z \\ \Gamma'_0 \end{bmatrix} \;\middle|\; \theta \in \mathbb{R} \right\}$$

where

$$H'_\nu := \left[\begin{array}{c|c} H_\nu & 0 \\ \hline 0 & 0 \end{array}\right], \qquad \Gamma'_0 := \left[\begin{array}{c|c} \Gamma_0 & q \\ \hline 0 & 0 \end{array}\right] \qquad q := \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

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Non-Unital Lindblad Equation
with $V_k := C_k + iD_k$ and $\{C_k, D_k\}_+ = 0$

$$\begin{aligned}\Gamma(\rho) &= \tfrac{1}{2}\sum_k V_k^\dagger V_k \rho + \rho V_k^\dagger V_k - 2 V_k \rho V_k^\dagger \\ &= \Big(\tfrac{1}{2}\sum_{k=1} \mathrm{ad}^2_{C_k} + \mathrm{ad}^2_{D_k} + 2i\, \mathrm{ad}_{C_k}\, \mathrm{ad}^+_{D_k}\Big)(\rho)\ ,\end{aligned}$$

Ex.: 1-qubit system $\mathfrak{g}_0 \subset \mathfrak{g} \subset \mathfrak{gl}(4, \mathbb{C})$:

- unital single-qubit channels

$$\mathfrak{g}_0 := \langle i\hat\sigma_\nu, \hat\sigma_\nu^2, \{\hat\sigma_\nu, \hat\sigma_\mu\}_+ \,|\, \nu \neq \mu \in \{x,y,z\}\rangle_{\mathrm{Lie}} \overset{\mathrm{iso}}{=} \mathfrak{gl}(3, \mathbb{R})$$

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- observe $\mathfrak{g} := \mathfrak{g}_0 \oplus_s \mathfrak{i}_0$, where

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by
$$\begin{aligned} [\mathfrak{g}_0, \mathfrak{g}_0] &\subseteq \mathfrak{g}_0 \\ [\mathfrak{g}_0, \mathfrak{i}_0] &\subseteq \mathfrak{i}_0 \\ [\mathfrak{i}_0, \mathfrak{i}_0] &= 0 \in \mathfrak{i}_0 \end{aligned}$$

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