Private quantum subsystems and error correction

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15 December 2014
QEC 2014 – Zurich

Work in collaboration with David Kribs, Raymond Laflamme, and Sarah Plosker

UNIVERSITY OF WATERLOO | IQC Institute for Quantum Computing

Last time at QEC...

On Complementarity In QEC And Quantum Cryptography

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Department of Mathematics & Statistics
University of Guelph

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Institute for Quantum Computing
University of Waterloo

QEC II — USC — December 2011

Outline

- Review definitions of operator quantum privacy and error correction
- Complementary between privacy and error correction
- Restrictions of operator quantum privacy
- Generalized notion of subsystem privacy
- Recovering the duality with quantum error correction

Notation

- Subsystems: $S = (A \otimes B) \oplus (A \otimes B)^{\perp}$
- Density matrices: Bounded linear operators with trace 1, $\sigma_A \in A,\ \sigma_B \in B,\ \rho \in A \otimes B$
- Quantum channel: Completely positive trace preserving map between linear operators, $\Phi : \mathcal{B}(A) \to \mathcal{B}(C)$
- Complementary channel: Given a quantum channel Φ, there always exists a unitary U_Φ and ancillary state $|\phi\rangle\langle\phi|_K$ such that $\Phi(\rho_A) = \mathrm{Tr}_K\left(U_\Phi(\rho_A \otimes |\phi\rangle\langle\phi|_K)U_\Phi^\dagger\right),\ \forall\ \rho_i$. The **complementary channel** is then defined as:

$$\Phi^\sharp(\rho) = \mathrm{Tr}_C\left(U_\Phi(\rho \otimes |\phi\rangle\langle\phi|_K)U_\Phi^\dagger\right).$$

Operator QEC and privacy

$$S = (A \otimes B) \oplus (A \otimes B)^{\perp}$$

- A subsystem B is an **operator private subsystem** for Φ if there exists ρ_0 such that

$$\Phi(\sigma_A \otimes \sigma_B) = \rho_0, \ \forall \sigma_A, \ \sigma_B$$

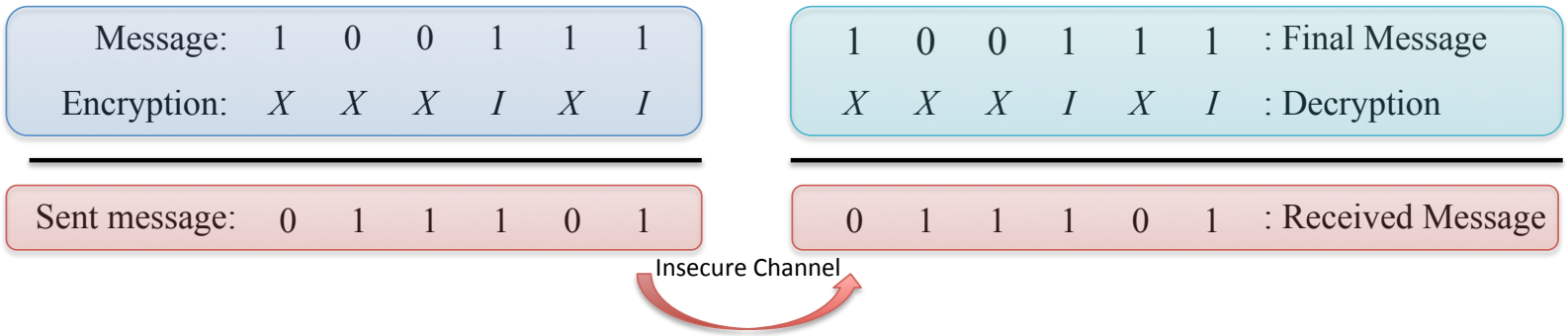
- A subsystem B is **operator quantum error correctable** for $\mathcal{E}$ if there exist $\tau_A(\sigma_A)$, $\mathcal{R}$ such that

$$\mathcal{R} \circ \mathcal{E}(\sigma_A \otimes \sigma_B) = \tau_A \otimes \sigma_B$$

Random Unitary channels

What type of channels are required to privatize quantum information?

In classical communication, messages can be encrypted using a one-time pad.



The key property of the one-time pad is the uniform *randomization* of each of the bits of the message.

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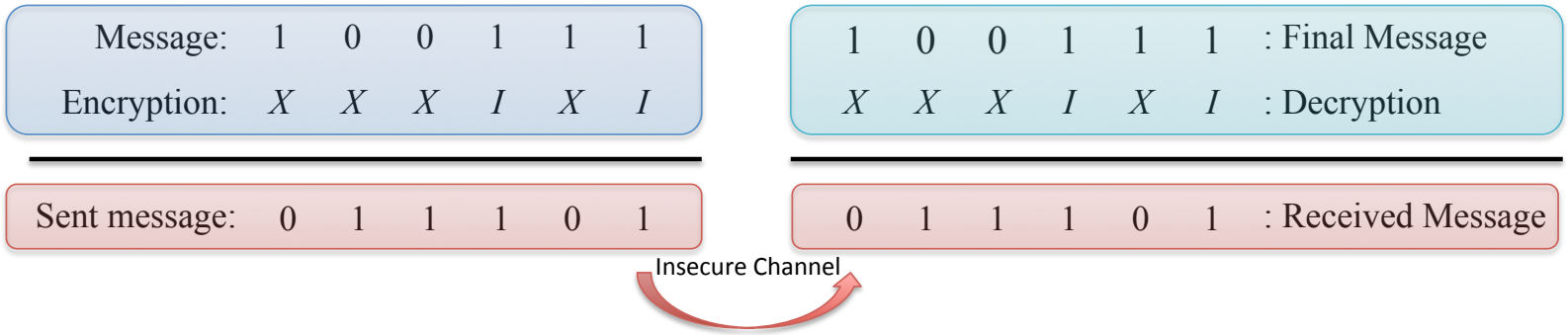

The key property of the one-time pad is the uniform *randomization* of each of the bits of the message.
The state of any given bit of encrypted data x_b is given by a classical probability distribution:

$$\Phi(x_b) = \frac{1}{2}x_b + \frac{1}{2}\overline{x_b}$$

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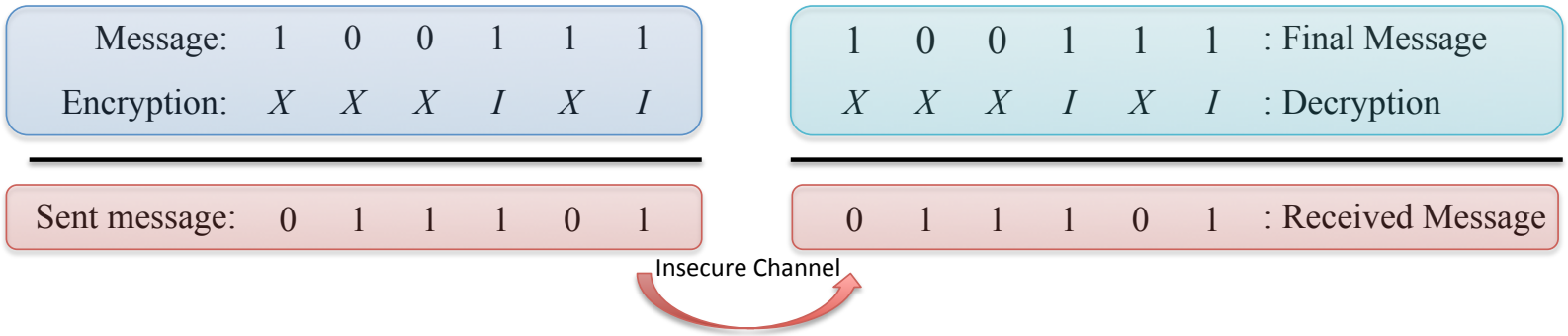


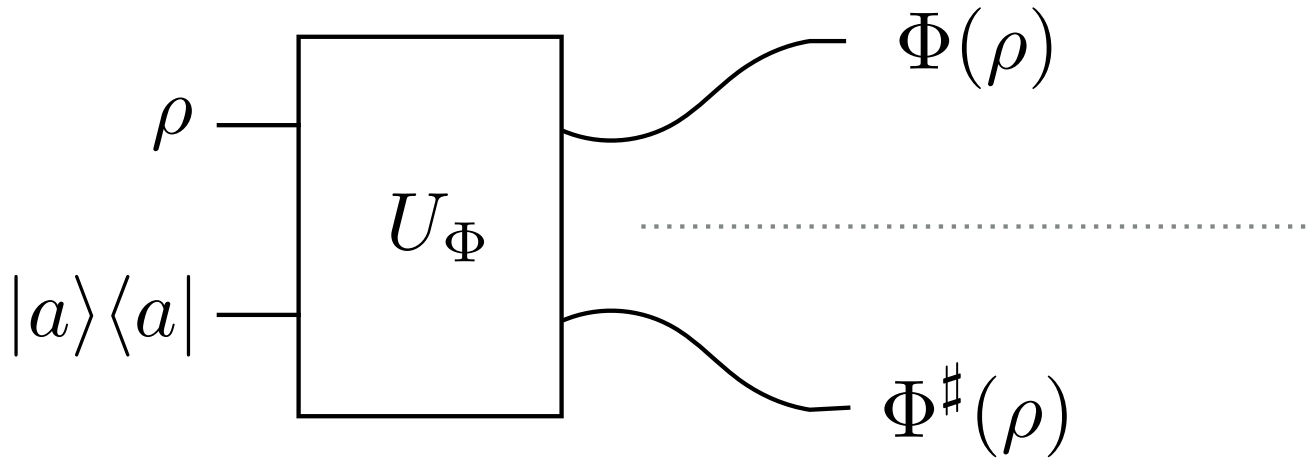
Random unitary channels provide the quantum analogue to the classical one-time pad,

$$\Phi(\rho) = \sum_i p_i U_i \rho U_i^\dagger$$

Operator duality

Theorem (KKS08[1])

A subsystem B is an operator private subsystem for a channel Φ if and only if it is operator QEC for the complementary channel $\Phi^\sharp$.



[1] D. Kretschmann, D. W. Kribs, R. Spekkens, (2008)

Quest for small private channels

Inspiration from quantum error correction!

The dephasing channel is not private on a single qubit:

$$\Lambda_i(\rho) = \frac{1}{2}(\rho + Z_i \rho Z_i) \qquad \forall \rho \in S.$$

How about the same identical channel on multiple qubits?

$$\Lambda(\rho) \quad = \Phi_2 \circ \Phi_1(\rho)$$

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The resulting mapping yields:

$$\begin{pmatrix} \alpha_{00} & \alpha_{01} & \alpha_{02} & \alpha_{03} \\ \alpha_{10} & \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{20} & \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{30} & \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \xrightarrow{\Lambda} \begin{pmatrix} \alpha_{00} & 0 & 0 & 0 \\ 0 & \alpha_{11} & 0 & 0 \\ 0 & 0 & \alpha_{22} & 0 \\ 0 & 0 & 0 & \alpha_{33} \end{pmatrix}$$

No-go result for private subspaces

Theorem (JKLP13[2])

Let $\Phi(\rho) = \sum_i p_i U_i \rho U_i^\dagger$ be a random unitary channel with mutually commuting Kraus operators. Then Φ has no private subspace.

[2] TJ, D. W. Kribs, R. Laflamme, S. Plosker (2013)

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- A subsystem B is an **operator private subsystem** for Φ if there exists ρ_0 such that

$$\Phi(\sigma_A \otimes \sigma_B) = \rho_0, \ \forall \sigma_A, \ \sigma_B$$

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- A subsystem B is an **operator private subsystem** for Φ if there exists ρ_0 such that

$$\Phi(\sigma_A \otimes |\psi\rangle\langle\psi|) = \rho_0, \ \forall \sigma_A, \ |\psi\rangle\langle\psi|$$

Therefore, the channel $\Lambda = \Lambda_2 \circ \Lambda_1$ **cannot** be operator quantum private

[2] TJ, D. W. Kribs, R. Laflamme, S. Plosker (2013)

However...

- Consider the following encoding of a quantum state:

$$\rho_L = \frac{1}{2}(I + \alpha XX + \beta YI + \gamma ZX).$$

  ρ_L is privatized by the channel $\Lambda = \Lambda_2 \circ \Lambda_1$. A contradiction?
- It can be shown that the state space defined by the parameters $\alpha,\ \beta,\ \gamma$ is unitarily equivalent to $I_2 \otimes \mathcal{D}_2$, where $\mathcal{D}_2$ is the space of 2-dimensional density matrices.

Where is the loophole?

Λ privatizes the state space $I_2 \otimes \mathcal{D}_2$, why is this not equivalent to operator privacy?
A subsystem B is an **operator private subsystem** for Φ if there exists ρ_0 such that

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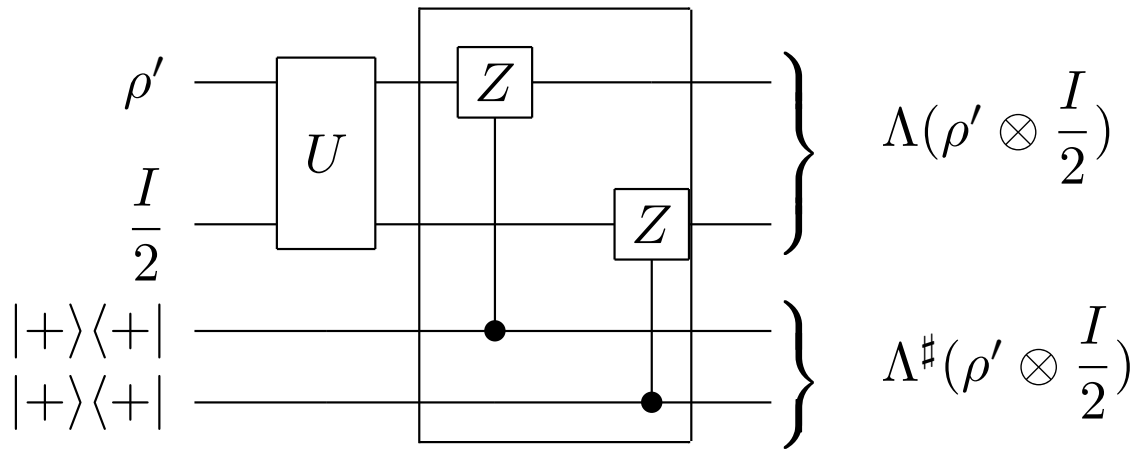
Therefore, fixing the state $\sigma_A = I_2$, is what allows the channel to be private, suggesting a new notion of privacy, **private quantum channels**.

The role of a fixed ancilla

- A subsystem B is a **private quantum subsystem**[3] for Φ if there is a $\rho_0 \in S$ and $\sigma_A \in A$ such that

$$\Phi(\sigma_A \otimes \sigma_B) = \rho_0, \ \forall \sigma_B \in B$$

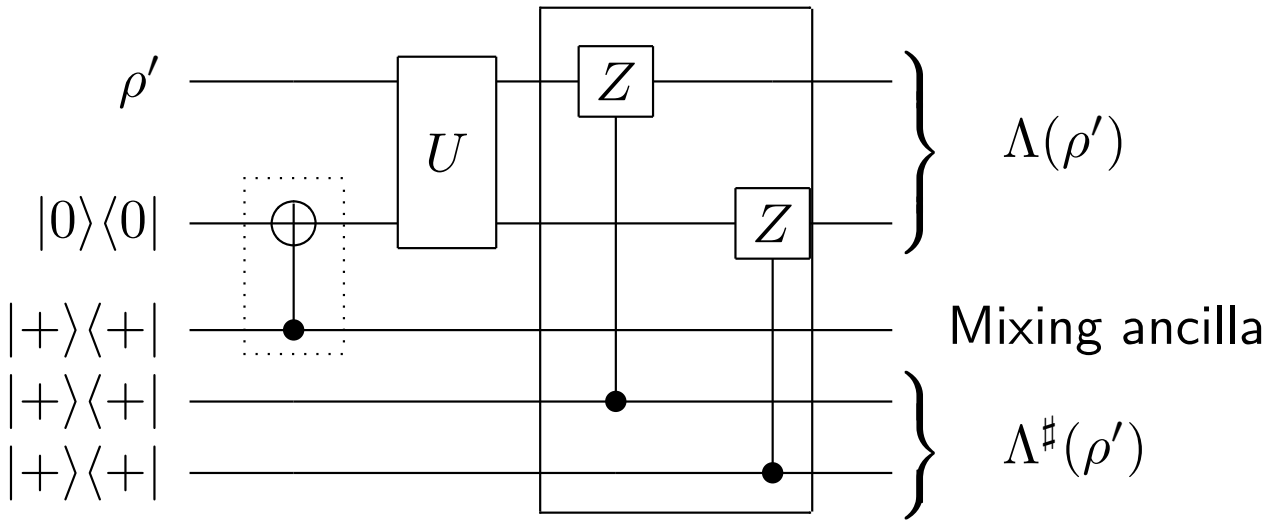
- The conjugate channel to the multi-qubit phase damping channel $\Lambda = \Lambda_2 \circ \Lambda_1$ cannot be operator quantum error correctable. In fact, it is private for the same encoding space.



[3] S. D. Bartlett, T. Rudolph, R. W. Spekkens (2004)

What happens to the duality with error correction?

The mixed state ancilla is the resource allowing for privacy of the channel.



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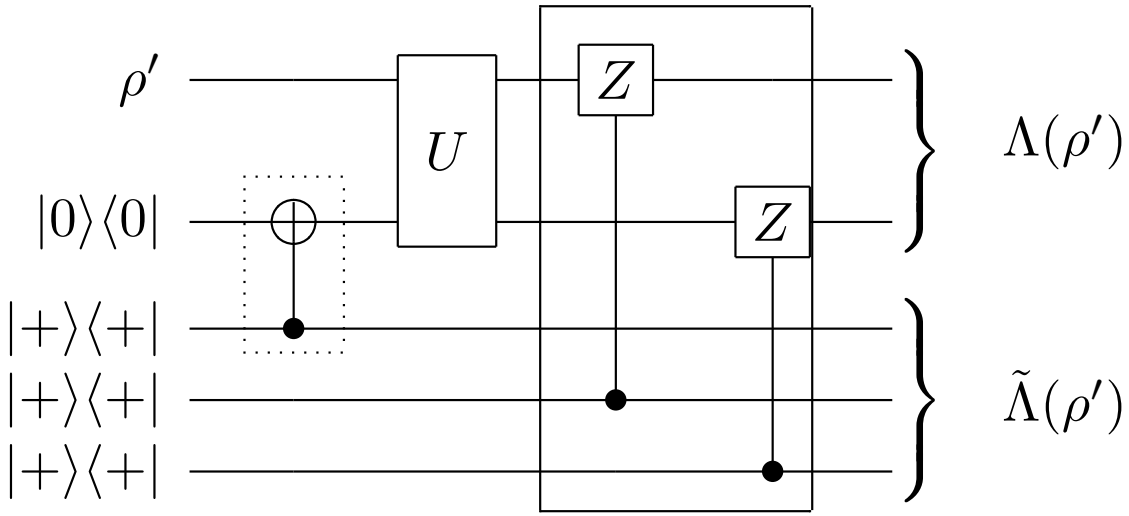

The generalized complementary channel $\tilde{\Lambda}$ must be quantum error correctable by the operator duality that exists on the extended Hilbert space.

Additional degrees of freedom

Operator

$$\alpha \left|0_L\right\rangle_{12} \rightarrow \sum_{ij} \left|ij\right\rangle_{12} \left|E^0_{ij}\right\rangle_K$$

$$\beta \left|1_L\right\rangle_{12} \rightarrow \sum_{ij} \left|ij\right\rangle_{12} \left|E^1_{ij}\right\rangle_K$$

Generalized

$$\alpha \left|0_L\right\rangle_{123} \rightarrow \sum_{ijk} \left|ijk\right\rangle_{123} \left|E^0_{ijk}\right\rangle_K$$

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$$\begin{aligned} |ij\rangle\langle kl|_{12} \mathsf{Tr}_E \Big(& |\alpha|^2 |E^0_{ij}\rangle\langle E^0_{kl}| \\ & + \alpha\beta^* |E^0_{ij}\rangle\langle E^1_{kl}| \\ & + \alpha^*\beta |E^1_{ij}\rangle\langle E^0_{kl}| \\ & + |\beta|^2 |E^1_{ij}\rangle\langle E^1_{kl}| \Big) \end{aligned}$$

$$\begin{aligned} \implies \langle E^0_{ij}|E^0_{kl}\rangle &= \langle E^1_{ij}|E^1_{kl}\rangle, \\ \langle E^0_{ij}|E^1_{kl}\rangle &= 0 \end{aligned}$$

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$$\begin{aligned}|ij\rangle\langle kl|_{12}\mathsf{Tr}_E\Big(&|\alpha|^2(|E^0_{ij0}\rangle\langle E^0_{kl0}| + |E^0_{ij1}\rangle\langle E^0_{kl1}|)\\ &+ \alpha\beta^*(|E^0_{ij0}\rangle\langle E^1_{kl0}| + |E^0_{ij1}\rangle\langle E^1_{kl1}|)\\ &+ \alpha^*\beta(|E^1_{ij0}\rangle\langle E^0_{kl0}| + |E^1_{ij1}\rangle\langle E^0_{kl1}|)\\ &+ |\beta|^2(|E^1_{ij0}\rangle\langle E^1_{kl0}| + |E^1_{ij1}\rangle\langle E^1_{kl1}|)\Big)\end{aligned}$$

$$\langle E^0_{ij0}|E^0_{kl0}\rangle + \langle E^0_{ij1}|E^0_{kl1}\rangle = \langle E^1_{ij0}|E^1_{kl0}\rangle + \langle E^1_{ij1}|E^1_{kl1}\rangle$$
$$\langle E^0_{ij0}|E^1_{kl0}\rangle = -\langle E^0_{ij1}|E^1_{kl1}\rangle$$

Additional degrees of freedom

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- Operator privacy:

$$\implies \langle E^0_{ij}|E^0_{kl}\rangle = \langle E^1_{ij}|E^1_{kl}\rangle,$$
$$\langle E^0_{ij}|E^1_{kl}\rangle = 0$$

- Generalized operator privacy:

$$\langle E^0_{ij0}|E^0_{kl0}\rangle + \langle E^0_{ij1}|E^0_{kl1}\rangle = \langle E^1_{ij0}|E^1_{kl0}\rangle + \langle E^1_{ij1}|E^1_{kl1}\rangle$$
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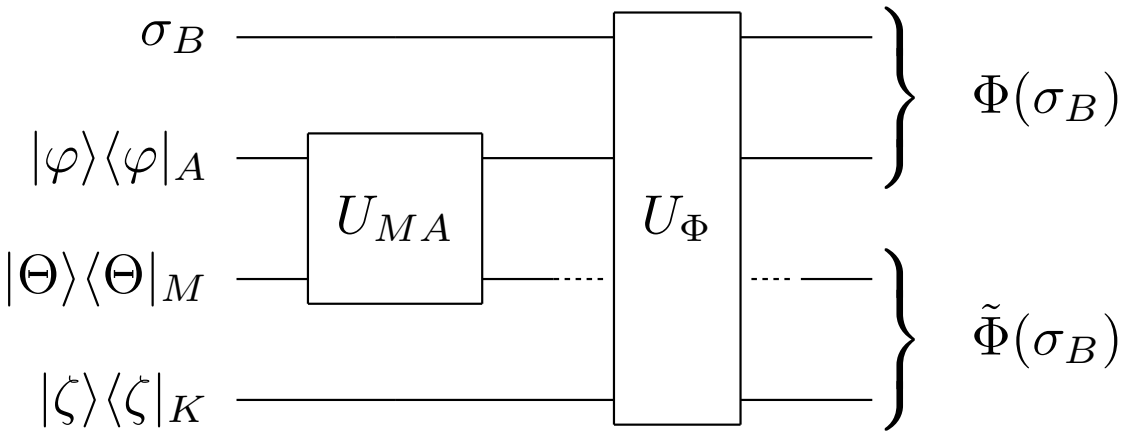
Generalized complementary channel



- By purifying the ancillary space, the duality between privacy and error correction is recovered for private subsystem channels!
- What about a generalized notion of error correction?
  A subsystem B is **generalized operator quantum error correctable** for $\mathcal{E}$ if there exists a channel $\mathcal{R}$, a fixed state σ_A, and a state τ_A such that

$$\mathcal{R} \circ \mathcal{E}(\sigma_A \otimes \sigma_B) = \tau_A \otimes \sigma_B, \ \forall \ \sigma_B$$

New notion of QEC?

A subsystem B is **generalized operator quantum error correctable (GenOQEC)** for $\mathcal{E}$ if there exists a channel $\mathcal{R}$, a fixed state σ_A, and a state τ_A such that

$$\mathcal{R} \circ \mathcal{E}(\sigma_A \otimes \sigma_B) = \tau_A \otimes \sigma_B, \ \forall \ \sigma_B$$

- There is no added benefit to the generalized notion of operator quantum error correction
- Given a GenOQEC channel Φ for a subsystem B with a fixed ancilla state $\sigma_A = \sum p_i |\psi_i\rangle\langle\psi_i|$ and output ancilla τ_A. Then, the channel is OQEC for any $|\psi_i\rangle\langle\psi_i|$[4]:

$$\implies B \text{ is OQEC for } \Phi$$

[4] TJ, Kribs, Laflamme, Plosker (2014)

Summary

- Private quantum channels provide a quantum analog to the classical one-time pad
- Random commuting unitary channels cannot yield operator private subsystems
- Encoding information into fixed subsystems provide additional freedom
- Duality between general private subsystems and error correction only recovered when extending the Hilbert space beyond standard complementarity

Thank you for your attention!

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