XS-Stabilizer

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Definition

- Pauli-S group: \( P_n^S = \langle \alpha, X, S \rangle^\otimes n \)

\[
\alpha = \sqrt{i} \quad S = \text{diag}(1, i) \quad S^{-1}XS = -iXZ
\]

- Given \( G = \langle g_1, \ldots, g_m \rangle \subset P_n^S \)

We call a state \( |\psi\rangle \) XS-stabilizer state if (uniquely)

\[
g_j |\psi\rangle = |\psi\rangle
\]

When not unique, we call it XS-stabilizer code
Outline

- Operator picture
- State picture
Operator picture
Starting point of Pauli stabilizer

- Either commute or anti-commute
- Each generator evenly split the Hilbert space
- Commutativity allows consecutively splitting
Example

$P(X \otimes S)$ is Hermitian

$X \otimes S$

$P \quad (X \otimes S)^2 = I \otimes Z$
Example

\[ i^3 S \otimes S \otimes S \]

Only one of \( x_1, x_2, x_3 \) is equal to 1

(Positive) 1-in-3 SAT problem

NP-Complete
Two operators

1. Commute and independent

2. Commute but not fully independent

\[ g_1 = X \otimes S \otimes I \]
\[ g_2 = I \otimes S \otimes X \]
\[ g_1^2 = I \otimes Z \otimes I = g_2^2 \]
Two operators

3. Partially commute

\[ g_1 = X \otimes X \otimes S \otimes S \quad \rightarrow \quad P_1 \]
\[ g_2 = S \otimes S \otimes X \otimes X \quad \rightarrow \quad P_2 \]

\[ g_1 g_2 g_1^{-1} g_2^{-1} = Z \otimes Z \otimes Z \otimes Z \]

\[ P_{12} = \frac{1}{2} (1 + Z \otimes Z \otimes Z \otimes Z) \]
Commuting projectors

\[ g_1 |\psi\rangle = g_2 |\psi\rangle = |\psi\rangle \]

\[ P_1 P_{12} |\psi\rangle = P_2 P_{12} |\psi\rangle = P_{12} |\psi\rangle = |\psi\rangle \]
Find codeword state

• Given $G = \langle g_1, \ldots, g_m \rangle \subset P_n^s$, define diagonal subgroup as $G_D$.

• We can construct a codeword state $|\psi_x\rangle$, if we can find a computational basis $|x\rangle$ stabilized by $G_D$.

• When $G_D$ is generated by Z-type operators, this procedure is efficient.
Diagonal subgroup

Each element of $G$ has the form: $\mathcal{Z} g_1^{x_1} \cdots g_m^{x_m}$, where $\mathcal{Z}$ is generated by \{g_j^2\} $\cup$ \{g_j g_k g_j^{-1} g_k^{-1}\}

So we can write down a set of generators of $G_D$ by using linear algebra.
Operator picture

• Properties of operators
• Computational complexity
• Equivalent commuting projectors
• Find code states
The state picture
The state picture

- Concrete
- Easiest way to utilize the uniqueness condition
- (Innsbruck-Munich influence)
Example

\[ g_1 = X \otimes S^3 \otimes S^3 \otimes S \otimes X \otimes X, \]
\[ g_2 = S^3 \otimes X \otimes S^3 \otimes X \otimes S \otimes X, \]
\[ g_3 = S^3 \otimes S^3 \otimes X \otimes X \otimes X \otimes S. \]

\[
\sum_{x_j=0}^{1} (-1)^{x_1x_2x_3} |x_1, x_2, x_3, x_2 \oplus x_3, x_1 \oplus x_3, x_1 \oplus x_2\rangle
\]
Mechanism

\[ \mathbb{Z} \otimes \mathbb{Z} \otimes \mathbb{Z} \]

\[ \sum_{x_1, x_2} |x_1, x_2, x_1 \oplus x_2 \rangle \]
Mechanism

\[ X \otimes Z \]

\[ |0, x_2\rangle \leftrightarrow (-1)^{x_2} |1, x_2\rangle \]

\[ \sum (-1)^{x_1 x_2} |x_1, x_2\rangle \]
Mechanism

\[ X \otimes S \otimes \cdots \]

\[ |0, x_2 \oplus x_3, \cdots \rangle \leftrightarrow i^{x_2+x_3} (-1)^{x_2 x_3} |1, x_2 \oplus x_3, \cdots \rangle \]

\[ \sum i^{x_1 (x_2 + x_3)} (-1)^{x_1 x_2 x_3} |x_1, x_2 \oplus x_3, \cdots \rangle \]

Bravyi, Haah 2012
Twisted quantum double

- Double semion: \[ \sum_{x \text{ is close loops}} (-1)^{\text{number of loops}} |x\rangle \]
Twisted quantum double

- twisted double on $\mathbb{Z}_2^n$

Flip a (plaquette) loop, add a quadratic phase

Hu, Wan, Wu 2012
Mechanism

\[ X \otimes S \otimes \cdots \]

\[ |0, x_2 \oplus x_3, \cdots \rangle \leftrightarrow i^{x_2+x_3} (-1)^{x_2 x_3} |1, x_2 \oplus x_3, \cdots \rangle \]

\[ \sum \left( i^{x_1(x_2+x_3)} (-1)^{x_1 x_2 x_3} |x_1, x_2 \oplus x_3, \cdots \rangle \right) \]
S-CZ gadget

\[ \sum_{x_1, x_2} |x_1, x_2, x_1 \oplus x_2\rangle \]

\[ S^{-1} \otimes S^{-1} \otimes S = CZ_{12} \otimes I \]
Why quadratic?

If we have \( X \otimes \sqrt{S} \otimes \cdots \)

\[(X \otimes \sqrt{S} \otimes \cdots)^2 = I \otimes S \otimes \cdots\]

Hard to make it compatible with the string intuition
Discussion

- Should we add $CZ$ to the Pauli-$S$ group?
- There’s some tradeoff. Choose what is the most convenient for you
Other funny facts

• XS states have very similar entanglement properties compared to Pauli states (~Flammia, Hamma, Hughes, Wen)

• Double semion (and probably other twisted double model) have transversal logical-X gate
Some error deserves not to be corrected

Thanks